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On the constructive determination of spectral invariants of the periodic Schrödinger operator with smooth potentials

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Abstract

In this paper, we develop the asymptotic formulae, obtained in my previous papers, for the band functions and the Bloch functions of the Schrödinger operator with the smooth periodic potentials. Then using these formulae, we determine constructively a family of spectral invariants of this operator from the given band functions. Some of these invariants generalize the well-known invariants and others are entirely new. The new invariants are explicitly expressed by Fourier coefficients of the potential which present the possibility of determining the potential constructively by using the band functions as input data.

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1. Introduction

We investigate the Schrödinger operator $L(q) = -\Delta + q$ in $L_2(\mathbb{R}^d), d \geqslant 2$, with a real periodic, relative to a lattice Ω in $\mathbb{R}^d$, potential $q \in W_2^s(F)$, where F is the fundamental domain $\mathbb{R}^d/\Omega$ of Ω, $s \geqslant 6(3^d(d+1)^2) + d$. The relation $q \in W_2^s(F)$ means that

$$q(x) = \sum_{\gamma\in\Gamma} q_\gamma \, e^{in\langle\gamma,x\rangle}, \qquad x \in \mathbb{R}^d, \qquad \text{and} \qquad \sum_{\gamma\in\Gamma} |q_\gamma|^2(1+|\gamma|^{2s}) < \infty, \tag{1}$$

where $\Gamma \equiv \{\delta \in \mathbb{R}^d : \langle\delta,\omega\rangle \in 2\pi\mathbb{Z}, \forall\,\omega \in \Omega\}$ is the lattice dual to Ω, $q_\gamma = \int_F q(x)\,e^{-i\langle\gamma,x\rangle}\,dx$ is the Fourier coefficient of q and $\langle\cdot,\cdot\rangle$ and $|\cdot|$ are the inner product and norm in $\mathbb{R}^d$. Without loss of generality, it can be assumed that the measure $\mu(F)$ of F is 1 and $q_0 = 0$. The spectrum of $L(q)$ is the union of the spectra of the operators $L_t(q)$, for $t \in F^* \equiv \mathbb{R}^d/\Gamma$, generated in $L_2(F)$ by the differential expression $-\Delta u + qu$ and the conditions

$$u(x+\omega) = e^{i\langle t,\omega\rangle} u(x), \qquad \forall\omega \in \Omega$$

(see [1, 5]). The eigenvalues $\Lambda_1(t) \leqslant \Lambda_2(t) \leqslant \cdots$ of $L_t(q)$ define functions $\Lambda_1(t), \Lambda_2(t), \ldots,$ of t that are called the band functions of $L(q)$. The eigenfunction $\Psi_{n,t}$ of $L_t(q)$ corresponding to $\Lambda_n(t)$ is known as the Bloch Function. The main purpose of this paper is the constructive determination of a family of spectral invariants from the given band functions.

In the introduction section, we present a list of the main results. In section 2, we prove the main results without giving some estimations which are given in section 3 and in appendices. To list the main results, the following notations are introduced. Let δ be a visible point of Γ, i.e., δ is the nonzero element of Γ of minimal norm belonging to the line $\delta\mathbb{R}$, $\Omega_\delta = \{h \in \Omega : \langle h, \delta\rangle = 0\}$ be a sublattice of Ω in the hyperplane $H_\delta = \{x \in \mathbb{R}^d : \langle x, \delta\rangle = 0\}$ and Γ_δ be a dual lattice of Ω_δ. Denote by $M(\Gamma)$ and $M(\Gamma_\delta)$ the set of all visible points of the lattices Γ and Γ_δ, respectively. The spectral invariants are expressed by the band functions and the Bloch functions of the Schrödinger operator $L(q^\delta)$ in $L_2(\mathbb{R}^d)$ with the directional potential

$$q^\delta(x) = \sum_{n\in\mathbb{Z}} q_{n\delta}\,\mathrm{e}^{in\langle\delta,x\rangle}, \qquad x \in \mathbb{R}^d \tag{2}$$

which is the restriction of the original potential q to the linear span of $\{\mathrm{e}^{in\langle\delta,x\rangle} : n \in \mathbb{Z}\}$. The function q^δ depends only on one variable $\zeta = \langle\delta, x\rangle$ and can be written as

$$q^\delta(x) = Q(\langle\delta, x\rangle), \qquad \text{where} \qquad Q(\zeta) = \sum_{n\in\mathbb{Z}} q_{n\delta}\,\mathrm{e}^{in\zeta}.$$

The band functions and the Bloch functions of the operator $L(q^\delta)$ are expressed by eigenvalues $\mu_j(v)$ and eigenfunctions $\varphi_{j,v}(\zeta)$ of the Sturm–Liouville operator $T_v(Q)$ generated by the boundary value problem

$$-|\delta|^2 y''(\zeta) + Q(\zeta)y(\zeta) = \mu y(\zeta), \qquad y(2\pi) = \mathrm{e}^{\mathrm{i}2\pi v}y(0), \qquad y'(2\pi) = \mathrm{e}^{\mathrm{i}2\pi v}y'(0),$$

where $j \in \mathbb{Z}$, $v \in [0, 1)$.

In the pioneering paper [2] about isospectral potentials, it was proven that if $q \in C^6(F), \omega \in \Omega\backslash 0$, and δ is the visible point of Γ satisfying $\langle\delta, \omega\rangle = 0$, then given band functions one may recover the eigenvalues of $T_v(Q)$ for $v = 0, \frac{1}{2}$ and the invariants $I(\omega, \delta, j, v)$ for $j \in \mathbb{Z}$, $v = 0, \frac{1}{2}$, where

$$I(\omega, \delta, j, v) = \int_F |Q_\omega(x)\varphi_{j,v}(\langle x, \delta\rangle)|^2\,\mathrm{d}x$$

if $\mu_j(v)$ is a simple eigenvalue,

$$I(\omega, \delta, j, v) = \int_F |Q_\omega(x)|^2((\varphi_{j+1,v}(\langle x, \delta\rangle))^2 + (\varphi_{j,v}(\langle x, \delta\rangle))^2)\,\mathrm{d}x$$

if $\mu_j(v)$ is not a simple eigenvalue, namely if $\mu_j(v) = \mu_{j+1}(v)$ and $Q_\omega(x)$ is defined by

$$Q_\omega(x) = \sum_{\gamma:\gamma\in\Gamma,\langle\gamma,\omega\rangle\neq 0} \frac{\gamma}{\langle\omega, \gamma\rangle} q_\gamma\,\mathrm{e}^{\mathrm{i}\langle\gamma,x\rangle}. \tag{3}$$

The proofs given in [2] were nonconstructive. In [3] a constructive way of determining the spectrum of $L_t(q^\delta)$ from the spectrum of $L_t(q)$ for the case $d = 2$ was given.

In this paper, we consider the Schrödinger operator $L(q)$ for arbitrary dimension d and using a given band function as input date, we constructively determine all eigenvalues of $T_v(Q)$ for all values of $v \in [0, 1)$ and a family of new spectral invariants

$$J(\delta, b, j, v) = \int_F |q_{\delta,b}(x)\varphi_{j,v}(\langle\delta, x\rangle)|^2\,\mathrm{d}x \tag{4}$$

for $\upsilon \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$, $j \in \mathbb{Z}$, and for all visible elements b and δ of Γ_δ and Γ, where

$$q_{\delta,b}(x) = \sum_{\gamma \in S(\delta,b)\backslash \delta\mathbb{R}} \frac{\gamma}{\langle b, \gamma\rangle} q_\gamma \, \mathrm{e}^{\mathrm{i}\langle \gamma, x\rangle}, \tag{5}$$

$S(\delta, b) = P(\delta, b) \cap \Gamma$, and $P(\delta, b)$ is the plane containing δ, b and 0. The formula (3) contains all Fourier coefficients q_γ of q except the Fourier coefficients corresponding to the vectors of a hyperplane. However, (5) contains only the Fourier coefficients corresponding to the vectors of the plane $P(\delta, b)$ except the vectors of $\delta\mathbb{R}$. If the potential q is a trigonometric polynomial and $d > 2$, then most of the polynomials (5) contain either two nonzero Fourier coefficients q_γ and $q_{-\gamma}$, where $q_{-\gamma} = \overline{q_\gamma}$, or four nonzero Fourier coefficients $q_{\gamma_1}, q_{\gamma_2}, q_{-\gamma_1}, q_{-\gamma_2}$, or 6 nonzero Fourier coefficients $q_{\gamma_1}, q_{\gamma_2}, q_{\gamma_3}, q_{-\gamma_1}, q_{-\gamma_2}, q_{-\gamma_3}$. Moreover $\mu_n(v)$, for $\upsilon \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$, $j \in \mathbb{Z}$, is a simple eigenvalue and the corresponding eigenfunction $\varphi_{n,v}(\zeta)$ has a simple asymptotic decomposition. Therefore, substituting the asymptotic decomposition

$$|\varphi_{n,v}(\zeta)|^2 = A_0 + \frac{A_1(\zeta)}{n} + \frac{A_2(\zeta)}{n^2} + \cdots, \tag{6}$$

where $A_k(\zeta)$ is expressed via $Q(\zeta)$, into (4) we find the new invariants

$$J_k(\delta, b) = \int_F |q_{\delta,b}(x)|^2 A_k(\langle \delta, x\rangle)\, \mathrm{d}x \tag{7}$$

for $k = 0, 1, 2, \ldots, \delta \in M(\Gamma), b \in M(\Gamma_\delta)$. Note that $J_k(\delta, b)$ is explicitly expressed by the Fourier coefficients of q. Moreover, if $d > 2$ and $q(x)$ is a trigonometric polynomial, then, in general, the number of the nonzero invariants (7) is greater than the number of nonzero Fourier coefficients of q and most of these invariants are explicitly expressed by m Fourier coefficients of q, where $m \leqslant 3$. This situation allows us to give (it will be given in next papers) an algorithm for finding the potential q from these spectral invariants.

Let us describe the brief scheme of the constructive determination of these invariants. We use the asymptotic formulae for the band function and the Bloch function obtained in [10]. Note that the similar asymptotic formulae have been obtained in [3, 4, 7–9]. First by improving the asymptotic formulae for the band functions and the Bloch functions, in the high energy region and near diffraction hyperplanes, obtained in [10], we get an asymptotic formula, where there are sharp estimations for the first and second terms of the asymptotic decomposition. To describe this improvement, let us introduce the following notations. The eigenvalues of the operator $L_t(0)$ with zero potential are $|\gamma + t|^2$ for $\gamma \in \Gamma$. If the quasimomentum $\gamma + t$ lies near the diffraction hyperplane

$$D_\delta = \{x \in \mathbb{R}^d : |x|^2 - |x + \delta|^2 = 0\}, \tag{8}$$

then the corresponding eigenvalue of $L_t(q)$ is close to the eigenvalue of the operator $L_t(q^\delta)$ with directional potential (2). To describe the eigenvalue of $L_t(q^\delta)$ we consider the lattice Γ_δ. Let $F_\delta \equiv H_\delta / \Gamma_\delta$ be the fundamental domain of Γ_δ. In this notation, the quasimomentum $\gamma + t$ has an orthogonal decomposition

$$\gamma + t = \beta + \tau + (j + v)\delta, \tag{9}$$

where $\beta \in \Gamma_\delta \subset H_\delta, \tau \in F_\delta \subset H_\delta, j \in \mathbb{Z}, v \in [0, 1)$ and v depends on β and t. The eigenvalues and eigenfunctions of the operator $L_t(q^\delta)$ are

$$\lambda_{j,\beta}(v, \tau) = |\beta + \tau|^2 + \mu_j(v), \qquad \Phi_{j,\beta}(x) = \mathrm{e}^{\mathrm{i}\langle \beta+\tau, x\rangle} \varphi_{j,v}(\zeta) \quad \text{for} \quad j \in \mathbb{Z}, \beta \in \Gamma_\delta. \tag{10}$$

We say that the large quasimomentum (9) lies near the diffraction hyperplane (8) if

$$\frac{1}{2}\rho < |\beta| < \frac{3}{2}\rho, \qquad j = O(\rho^{\alpha_1}), \qquad \alpha_k = 3^k \alpha, \qquad \alpha = \frac{1}{4(3^d(d+1))}, \tag{11}$$

where ρ is a large parameter and $k = 1, 2, \ldots, d$. In this paper, we construct a set of the quasimomentum near the diffraction plane D_δ such that if $\beta + \tau + (j + v)\delta$ (see (9)) belongs to this set, then there exists a unique eigenvalue, denoted by $\Lambda_{j,\beta}(v,\tau)$, of $L_t(q)$ satisfying

$$\Lambda_{j,\beta}(v,\tau) = \lambda_{j,\beta}(v,\tau) + O(\rho^{-a}), \tag{12}$$

$$\Lambda_{j,\beta}(v,\tau) = \lambda_{j,\beta}(v,\tau) + \frac{1}{4}\int_F \left|f^2_{\delta,\beta+\tau}\right| |\varphi_{j,v}|^2 \,\mathrm{d}x + O(\rho^{-3a+2\alpha_1}\ln\rho), \tag{13}$$

where $a = 1 - \alpha_d + \alpha$ and

$$f_{\delta,\beta+\tau}(x) = \sum_{\gamma:\gamma\in\Gamma\backslash\delta\mathbb{R},|\gamma|<\rho^\alpha} \frac{\gamma}{\langle\beta+\tau,\gamma\rangle} q_\gamma\, \mathrm{e}^{\mathrm{i}\langle\gamma,x\rangle}.$$

This is a simple eigenvalue and the corresponding eigenfunction $\Psi_{j,\beta}(x)$ satisfies

$$\Psi_{j,\beta}(x) = \Phi_{j,\beta}(x) + O(\rho^{-a}). \tag{14}$$

The remainders of the formulae (12), (13), (14) are $O(\rho^{-a})$, $O(\rho^{-3a+2\alpha_1}\ln\rho)$ and $O(\rho^{-a})$, respectively, while the remainders of the corresponding formulae, obtained in [10], are $O(\rho^{-\alpha_2})$, $O(\rho^{-2\alpha_2}(\ln\rho)^4)$ and $O(\rho^{-\alpha_2}\ln\rho)$ (see (3.39, (3.52) and (6.23) of [10]), where $a > 1 - \frac{1}{4(d+1)}$ and $-3a + 2\alpha_1 < -2$, but α_2 is a small number (see (11)). Moreover, the second term of (13) has an explicit and a suitable form for the constructive determination of new invariants. Besides, we prove that the derivative of $\Lambda_{j,\beta}(v,\tau)$ in the direction of $h = \frac{\beta+\tau}{|\beta+\tau|}$ satisfies

$$|\beta+\tau|\frac{\partial\Lambda_{j,\beta}(v,\tau)}{\partial h} = |\beta+\tau|^2 + O(\rho^{2-2a}) \tag{15}$$

and the derivative of the other simple eigenvalues, neighboring with $\Lambda_{j,\beta}(v,\tau)$, does not satisfy (15). Using these formulae, we constructively determine the eigenvalues of T_v for $v \in [0, 1)$ and the invariants (4), (7). Then, using the asymptotic formulae for the eigenvalues and the eigenfunctions of T_v, we find $A_0(\zeta)$, $A_1(\zeta)$ and $A_2(\zeta)$ (see (7)) and the invariants

$$\int_F |q_{\delta,b}(x)|^2 q^\delta(x)\,\mathrm{d}x, \tag{16}$$

$$\int_F |q^\delta(x)|^2\,\mathrm{d}x \tag{17}$$

(see appendix D). If the potential q is a trigonometric polynomial, then most of the directional potentials have the form

$$q^\delta(x) = q_\delta\,\mathrm{e}^{\mathrm{i}\langle\delta,x\rangle} + q_{-\delta}\,\mathrm{e}^{-\mathrm{i}\langle\delta,x\rangle}. \tag{18}$$

In this case, by direct calculations, we show that (see appendix D)

$$A_0 = 1, \qquad A_1 = 0, \qquad A_2 = \frac{q^\delta(x)}{2} + a_1|q_\delta|^2, \qquad A_3 = a_2 q^\delta(x) + a_3|q_\delta|^2,$$
$$A_4 = a_4 q^\delta(x) + a_5\left(q_\delta^2\,\mathrm{e}^{\mathrm{i}2\langle\delta,x\rangle} + q_{-\delta}^2\,\mathrm{e}^{-\mathrm{i}2\langle\delta,x\rangle}\right) + a_6, \tag{19}$$

where $a_1, a_2, \ldots, a_6$ are the known constants. Moreover using (19), (17), and (7) for $k = 2, 4$, we find the invariant

$$\int_F |q_{\delta,b}(x)|^2\left(q_\delta^2\,\mathrm{e}^{\mathrm{i}2\langle\delta,x\rangle} + q_{-\delta}^2\,\mathrm{e}^{-\mathrm{i}2\langle\delta,x\rangle}\right)\mathrm{d}x \tag{20}$$

in case (18). In a subsequent paper, we give an algorithm for finding the potential q by the invariants (16), (17), and (20).

2. The proofs of the main results

In this section, we give the proofs of the main results without getting the technical details. The technical details, namely the proof of the lemmas and some estimations, are given in section 3 and in appendices , respectively. To obtain the asymptotic formulae for large eigenvalues, we introduce a large parameter ρ. If the considered eigenvalue is of order ρ^2, then we write the potential $q \in W_2^s(F)$ in the form

$$q(x) = \sum_{\gamma\in\Gamma(\rho^\alpha)} q_\gamma \,\mathrm{e}^{\mathrm{i}\langle\gamma,x\rangle} + O(\rho^{-p\alpha}), \tag{21}$$

where $p = s - d$, $\Gamma(\rho^\alpha) = \{\gamma \in \Gamma : 0 < |\gamma| < \rho^\alpha\}$ and α is defined in (11). Note that the relation $q \in W_2^s(F)$, which means that (1) holds, implies that if $s \geqslant d$, then

$$\sum_{\gamma\in\Gamma} |q_\gamma| < c_1, \qquad \sup\left|\sum_{\gamma\notin\Gamma(\rho^\alpha)} q_\gamma \,\mathrm{e}^{\mathrm{i}\langle\gamma,x\rangle}\right| \leqslant \sum_{|\gamma|\geqslant\rho^\alpha} |q_\gamma| = O(\rho^{-p\alpha}), \tag{22}$$

i.e., (21) holds. Here and in subsequent estimations, we denote by c_i $(i = 1, 2, \ldots)$ the positive, independent of ρ, constants.

First, let us describe some results of [10] that we use in this paper. In [10], we constructed a set B_δ, which is called a simple set near the diffraction plane D_δ (see (8)), such that if the quasimomentum $\gamma + t = \beta + \tau + (j + v)\delta$ (see (9)) belongs to the simple set B_δ, then there exists a unique eigenvalue Λ_N, which is simple, of $L_t(q)$ satisfying

$$|\Lambda_N - E(\lambda_{j,\beta}(v, \tau))| < \varepsilon_1$$

(see theorem 6.1 and formula (6.2) of [10]), where $\varepsilon_1 = \rho^{-d-2\alpha}$ and $E(\lambda_{j,\beta}(v, \tau))$ is called the known part of Λ_N. Besides, we prove that all other eigenvalues of the operator $L_t(q)$ lie in the ε_1 neighborhood of the numbers $F(\gamma' + t)$ and $\lambda_j(\gamma' + t)$, where $\gamma' \in \Gamma$, which are called as the known parts of the other eigenvalues. In order that Λ_N does not coincide with the other eigenvalues, we use the following two simplicity conditions

$$|E(\lambda_{j,\beta}(v, \tau)) - F(\gamma' + t)| \geqslant 2\varepsilon_1, \qquad |E(\lambda_{j,\beta}(v, \tau)) - \lambda_i(\gamma' + t)| \geqslant 2\varepsilon_1. \tag{S.C.}$$

Briefly, B_δ is the set of $\beta + \tau + (j + v)\delta$ satisfying (S.C.). Thus we constructed the set B_δ by eliminating the set of quasimomenta $\gamma + t \equiv \beta + \tau + (j + v)\delta$ for which the known part $E(\lambda_{j,\beta}(v, \tau))$ of the corresponding eigenvalue is situated from the known parts of the other eigenvalues at a distance less than $2\varepsilon_1$. Then, we investigated the set B_δ. By (9), every vector w of $\mathbb{R}^d$ has decomposition $w = \gamma + t \equiv \beta + \tau + (j + v)\delta$, where $\beta \in \Gamma_\delta$, $\tau \in F_\delta$, $j \in \mathbb{Z}$, $v \in [0, 1)$. In [10] (see the formula (6.45), theorems 3.1 and 6.1 of [10]), we proved that if

$$j \in S_1(\rho), \qquad \beta \in S_2(\rho), \qquad v \in S_3(\beta, \rho), \qquad \tau \in S_4(\beta, j, v, \rho), \tag{23}$$

then $\beta + \tau + (j + v)\delta \in B_\delta$ and hence there exists a unique eigenvalue Λ_N, which is simple, of $L_t(q)$ satisfying

$$\Lambda_N = \lambda_{j,\beta}(v, \tau) + O(\rho^{-\alpha_2}) \tag{24}$$

and the corresponding eigenfunction $\Psi_{N,t}(x)$ satisfies

$$\Psi_{N,t}(x) = \Phi_{j,\beta}(x) + O(\rho^{-\alpha_2} \ln \rho), \tag{25}$$

where α_2 is defined in (11) and the set S_1, S_2, S_3, S_4 are defined as follows:

$$S_1(\rho) = \left\{ j \in \mathbb{Z} : |j| < \frac{\rho^{\alpha_1}}{2|\delta|^2} - \frac{3}{2} \right\},$$

$$S_2(\rho) = \left\{ \beta \in \Gamma_\delta : \beta \in \left(R_\delta\left(\frac{3}{2}\rho - d_\delta - 1\right) \Big\backslash R_\delta\left(\frac{1}{2}\rho + d_\delta + 1\right)\right) \Big\backslash \left(\bigcup_{b\in\Gamma_\delta(\rho^{\alpha_d})} V_b^\delta\left(\rho^{\frac{1}{2}}\right)\right) \right\}, \tag{26}$$

where $d_\delta = \sup_{x,y\in F_\delta}|x-y|$, $R_\delta(c) = \{x \in H_\delta : |x| < c\}$, $\Gamma_\delta(c) = \{b \in \Gamma_\delta : 0 < |b| < c\}$,

$$V_b^\delta(c) = \{x \in H_\delta : ||x+b|^2 - |x|^2| < c\}.$$

For $\beta \in S_2(\rho)$, the set $S_3(\beta,\rho)$ is defined by

$$S_3(\beta,\rho) = W(\rho)\backslash A(\beta,\rho), \tag{27}$$

where $W(\rho) \equiv \{v \in (0,1) : |\mu_j(v) - \mu_{j'}(v)| > \frac{2}{\ln\rho}, \forall j', j \in \mathbb{Z}, j' \neq j\}$,

$$A(\beta,\rho) = \bigcup_{b\in\Gamma_\delta(\rho^{\alpha_d})} A(\beta,b,\rho),$$

and $A(\beta,b,\rho) = \{v \in [0,1) : \exists j \in \mathbb{Z}, |2\langle\beta,b\rangle + |b|^2 + |(j+v)\delta|^2| < 4d_\delta\rho^{\alpha_d}\}$.

For $j \in S_1(\rho)$, $\beta \in S_2(\rho)$, $v \in S_3(\beta,\rho)$, the set $S_4(\beta,j,v,\rho)$ is the set of $\tau \in F_\delta$ for which $\beta+\tau+(j+v)\delta \in B_\delta$. In other words, $S_4(\beta,j,v,\rho)$ is the set of $\tau \in F_\delta$ for which $E(\lambda_{j,\beta}(v,\tau))$ satisfies the simplicity conditions (S.C.). Since the functions taking part in (S.C.) are measurable, $S_4(\beta,j,v,\rho)$ is a measurable set. In [10] (see (6.48) of [10]), we proved that

$$\mu(S_4(\beta,j,v,\rho)) = \mu(F_\delta)(1+O(\rho^{-\alpha})). \tag{28}$$

Remark 1. If (23) holds, then there exists a unique index $N(j,\beta,v,\tau)$, depending on j,β,v,τ, for which the eigenvalue $\Lambda_{N(j,\beta,v,\tau)}(t)$ satisfies (24). Instead of $N(j,\beta,v,\tau)$, we write $N(j,\beta)$ (or N) if v,τ (or j,β,v,τ) are unambiguous. In the asymptotic formulae (12)–(15), instead of $\Lambda_{N(j,\beta,v,\tau)}$ and $\Psi_{N(j,\beta,v,\tau),t}(x)$, we write $\Lambda_{j,\beta}(v,\tau)$ and $\Psi_{j,\beta}(x)$, respectively, in order to underline that $\Lambda_{j,\beta}(v,\tau)$ and $\Psi_{j,\beta}(x)$ are close to $\lambda_{j,\beta}(v,\tau)$ and $\Phi_{j,\beta}(x)$, where $\lambda_{j,\beta}(v,\tau)$ and $\Phi_{j,\beta}(x)$ are defined in (10).

To prove the asymptotic formulae (12)–(14), which are suitable for the constructive determination of the spectral invariants, we put an additional condition on β, namely, we suppose that

$$\beta \notin \bigcup_{b\in\Gamma_\delta(p\rho^\alpha)} V_b^\delta(\rho^a), \tag{29}$$

where $V_b^\delta(\rho^a)$ and $\Gamma_\delta(p\rho^\alpha)$ are defined in (26). By definition of $V_b^\delta(\rho^a)$, the relation (29) yields

$$||\beta|^2 - |\beta+\beta_1|^2| \geqslant \rho^a, \qquad \forall \beta_1 \in \Gamma_\delta(p\rho^\alpha). \tag{30}$$

Using the inequalities $|\beta_1| < p\rho^\alpha$, $|\tau| < d_\delta$, $a > 2\alpha$ (see (13)), we obtain

$$||\beta+\tau|^2 - |\beta+\beta_1+\tau|^2| > \tfrac{8}{9}\rho^a, \qquad \forall \beta_1 \in \Gamma_\delta(p\rho^\alpha). \tag{31}$$

Now we prove (12) by using (23), (31), and the following relation

$$(\Lambda_N(t) - \lambda_{j,\beta})b(N,j,\beta) = (\Psi_{N,t}, (q-q^\delta)\Phi_{j,\beta}), \tag{32}$$

which can be obtained from $L_t(q)\Psi_{n,t}(x) = \Lambda_n(t)\Psi_{n,t}(x)$ by multiplying by $\Phi_{j,\beta}(x)$ and using $L_t(q^\delta)\Phi_{j,\beta}(x) = \lambda_{j,\beta}\Phi_{j,\beta}(x)$, where $b(N,j,\beta) = (\Psi_{N,t},\Phi_{j,\beta})$ and $(\cdot,\cdot)$ is the inner product in $L_2(\mathbb{R}^d)$. In [10], using (21), we proved that (see (3.22) and (3.23) of [10]) if $|j\delta| < r$, $|\beta| > \frac{1}{2}\rho$, where $r \geqslant r_1$ and $r_1 = \frac{\rho^{\alpha_1}}{2|\delta|} + 2|\delta|$, then the following decomposition

$$(q(x) - q^\delta(x))\Phi_{j,\beta}(x) = \sum_{(j_1,\beta_1)\in Q(\rho^\alpha,9r)} A(j,\beta,j+j_1,\beta+\beta_1)\Phi_{j+j_1,\beta+\beta_1}(x) + O(\rho^{-p\alpha}) \tag{33}$$

of $(q(x) - q^{\delta}(x))\Phi_{j,\beta}(x)$ by eigenfunction of $L_t(q^{\delta})$ holds, where

$Q(\rho^{\alpha}, 9r) = \{(j, \beta) : |j\delta| < 9r, 0 < |\beta| < \rho^{\alpha}\}$ and

$$\sum_{(j_1,\beta_1)\in Q(\rho^{\alpha},9r)} |A(j, \beta, j + j_1, \beta + \beta_1)| < c_2. \tag{34}$$

Using this decomposition in (32), we get

$$(\Lambda_N(t) - \lambda_{j,\beta})b(N, j, \beta) = O(\rho^{-p\alpha}) + \sum_{(j_1,\beta_1)\in Q(\rho^{\alpha},9r)} A(j, \beta, j + j_1, \beta + \beta_1)b(N, j + j_1, \beta + \beta_1). \tag{35}$$

Remark 2. If $|j'\delta| < r$, $|\beta'| > \frac{1}{2}\rho$ and $|\Lambda_N - \lambda_{j',\beta'}| > c(\rho)$, then by (35) we have

$$b(N, j', \beta') = \sum_{(j_1,\beta_1)\in Q(\rho^{\alpha},9r)} \frac{A(j', \beta', j' + j_1, \beta' + \beta_1)b(N, j' + j_1, \beta' + \beta_1)}{\Lambda_N - \lambda_{j',\beta'}} + O\left(\frac{1}{\rho^{p\alpha}c(\rho)}\right).$$

If $j \in S_1(\rho)$, then $|j\delta| < r_1 = O(\rho^{\alpha_1})$ and in (35) instead of r we take r_1.

Theorem 1. *If (23) and (29) hold, then there exists a unique eigenvalue $\Lambda_{j,\beta}(v, \tau)$, which is simple, of $L_t(q)$ satisfying (12).*

Proof. Since there exists a unique eigenvalue $\Lambda_N(t)$ satisfying (24) and the corresponding eigenfunction satisfies (25) (see remark 1), we have $b(N, j, \beta) = 1 + O(\rho^{-\alpha_2} \ln \rho)$. Therefore, we need to prove that the right-hand side of (35) is $O(\rho^{-a})$. First, we show that

$$b(N, j + j_1, \beta + \beta_1) = O(\rho^{-a}) \tag{36}$$

for $\beta_1 \in \Gamma_\delta(p\rho^{\alpha})$, $j = o(\rho^{\frac{a}{2}})$, $j_1 = o(\rho^{\frac{a}{2}})$. For this, we prove the inequality

$$|\Lambda_N(t) - \lambda_{j+j_1,\beta+\beta_1}| > \tfrac{1}{2}\rho^{a}, \qquad \forall \beta_1 \in \Gamma_\delta(p\rho^{\alpha}), \qquad \forall j = o\big(\rho^{\frac{a}{2}}\big), \qquad \forall j_1 = o\big(\rho^{\frac{a}{2}}\big) \tag{37}$$

and use the formula

$$b(N, j + j_1, \beta_1 + \beta) = \frac{(\Psi_{N,t}, (q - q^{\delta})\Phi_{j+j_1,\beta_1+\beta})}{\Lambda_N - \lambda_{j+j_1,\beta_1+\beta}}, \tag{38}$$

which can be obtained from (32) by replacing the indices j, β with $j + j_1, \beta + \beta_1$. By (24), the inequality (37) holds if

$$|\mu_j(v) + |\beta + \tau|^2 - \mu_{j+j_1}(v) - |\beta + \beta_1 + \tau|^2| > \tfrac{5}{9}\rho^{a}.$$

This inequality can easily be obtained by using (31), the equalities $j = o(\rho^{\frac{a}{2}})$, $j + j_1 = o(\rho^{\frac{a}{2}})$ (see the conditions on j, j_1 in (36), (37)) and the formula

$$\mu_n(v) = |(n + v)\delta|^2 + O(n^{-1}) \tag{39}$$

(see [1, 6]). Note that the set of the eigenvalues of $T_v(0)$ with zero potential is a sequence $\{|(n+v)\delta|^2 : n \in \mathbb{Z}\}$ and it is not hard to see that the set of the eigenvalues of T_v can be written as a sequence $\{\mu_n(v) : n \in \mathbb{Z}\}$ satisfying (39). Thus (36) is proved. Using (36), the definition of $Q(\rho^{\alpha}, 9r_1)$, and the relations $r_1 = O(\rho^{\alpha_1})$ (see remark 2), $\alpha_1 < \frac{a}{2}$ (see (11), (13)), we obtain that all multiplicands $b(N, j + j_1, \beta + \beta_1)$ in the right-hand side of (35), in the case $r = r_1$, are $O(\rho^{-a})$. Hence, (34) implies that the right-hand side of (35) is $O(\rho^{-a})$. □

To prove the asymptotic formula (13), we iterate (35) , in the case $r = r_1$, as follows. If $|j\delta| < r_1$, then the summation in (35) is taken under condition $(j_1, \beta_1) \in Q(\rho^{\alpha}, 9r_1)$ (see remark 2). By the definition of $Q(\rho^{\alpha}, 9r_1)$, we have $|j_1\delta| < 9r_1$. Hence, $|(j + j_1)\delta| < r_2$, where $r_2 = 10r_1$. Therefore, using (37) and remark 2, we get

$$b(N, j + j_1, \beta_1 + \beta) = \sum_{(j_2,\beta_2)\in Q(\rho^{\alpha},9r_2)} \frac{A(j(1), \beta(1), j(2), \beta(2))b(N, j(2), \beta(2))}{\Lambda_N - \lambda_{j+j_1,\beta+\beta_1}} + O(\rho^{-p\alpha}),$$

where $j(k) = j + j_1 + j_2 + \cdots + j_k$, $\beta(k) = \beta + \beta_1 + \beta_2 + \cdots + \beta_k$ for $k = 0, 1, 2, \ldots$. Using this in (35), we obtain

$$(\Lambda_N - \lambda_{j,\beta})b(N, j, \beta) = O(\rho^{-p\alpha}) + \sum_{\substack{(j_1,\beta_1)\in Q(\rho^{\alpha},9r_1)\\ (j_2,\beta_2)\in Q(\rho^{\alpha},9r_2)}} \frac{A(j, \beta, j(1), \beta(1))A(j(1), \beta(1), j(2), \beta(2))b(N, j(2), \beta(2))}{\Lambda_N - \lambda_{j+j_1,\beta+\beta_1}}. \quad (40)$$

To prove (13), we use this formula and the following lemma.

Lemma 1. *Suppose (23) and (29) hold. If* $j' \neq j$, $|j'\delta| < r$, *where* $r = O(\rho^{\frac{1}{2}\alpha_2})$, $r \geqslant r_1$, *and* $r_1 = \frac{\rho^{\alpha_1}}{2|\delta|} + 2|\delta|$, *then* $b(N(j, \beta), j', \beta) = O(\rho^{-2a}r^2 \ln \rho)$.

Theorem 2. *If (23) and (29) hold, then there exists a unique eigenvalue* $\Lambda_{j,\beta}(v, \tau)$, *which is simple, of* $L_t(q)$ *satisfying (13).*

Proof. We prove this by using (40). To estimate the summation in the right-hand side of (40), we divide the terms in this summation into three groups. The terms of the first, second and third groups are the terms with multiplicands $b(N, j, \beta)$, $b(N, j(2), \beta)$ with $j(2) \neq j$, and $b(N, j(2), \beta(2))$ with $\beta(2) \neq \beta$, respectively. The sum of the terms of the first group is $C_1(\Lambda_N)b(N, j, \beta)$, where

$$C_1(\Lambda_N) = \sum_{(j_1,\beta_1)\in Q(\rho^{\alpha},9r_1)} \frac{A(j, \beta, j + j_1, \beta + \beta_1)A(j + j_1, \beta + \beta_1, j, \beta)}{\Lambda_N - \lambda_{j+j_1,\beta+\beta_1}}. \quad (41)$$

The sum of the terms of the second group is

$$\sum_{\substack{(j_1,\beta_1)\in Q(\rho^{\alpha},9r_1)\\ (j_2,\beta_2)\in Q(\rho^{\alpha},9r_2)}} \frac{A(j, \beta, j + j_1, \beta + \beta_1)A(j + j_1, \beta + \beta_1, j(2), \beta)}{\Lambda_N - \lambda_{j+j_1,\beta+\beta_1}} b(N, j(2), \beta),$$

where $j(2) \neq j$. Since $r_2 = 10r_1 = O(\rho^{\alpha_1})$ (see remark 2), the conditions on j, j_1, j_2 and lemma 1 imply that $j(2) = O(\rho^{\alpha_1})$ and $b(N, j(2), \beta) = O(\rho^{-2a+2\alpha_1} \ln \rho)$. Using this, (34) and (37), we obtain that the sum of the terms of the second group is $O(\rho^{-3a+2\alpha_1} \ln \rho)$. The sum of the terms of the third group is

$$\sum_{\substack{(j_1,\beta_1)\in Q(\rho^{\alpha},9r_1)\\ (j_2,\beta_2)\in Q(\rho^{\alpha},9r_2)}} \frac{A(j, \beta, j(1), \beta(1))A(j(1), \beta(1), j(2), \beta(2))}{\Lambda_N - \lambda_{j+j_1,\beta+\beta_1}} b(N, j(2), \beta(2)), \quad (42)$$

where $\beta(2) \neq \beta$. Using (37) and remark 2, we get

$$b(N, j(2), \beta(2)) = \sum_{(j_3,\beta_3)\in Q(\rho^{\alpha},9r_3)} \frac{A(j(2), \beta(2), j(3), \beta(3))b(N, j(3), \beta(3))}{\Lambda_N - \lambda_{j(2),\beta(2)}} + O(\rho^{-p\alpha}),$$

where $r_3 = 10r_2$. Substituting it into (42) and isolating the terms with multiplicands $b(N, j, \beta)$, we see that the sum of the terms of the third group is

$C_2(\Lambda_N)b(N, j, \beta) + C_3(\Lambda_N) + O(\rho^{-p\alpha})$, where

$$C_2(\Lambda_N) = \sum_{\substack{(j_1,\beta_1)\in Q(\rho^\alpha, 9r_1),\\ (j_2,\beta_2)\in Q(\rho^\alpha, 90r_1)}} \frac{A(j,\beta,j(1),\beta(1))A(j(1),\beta(1),j(2),\beta(2))A(j(2),\beta(2),j,\beta)}{(\Lambda_N-\lambda_{j+j_1,\beta+\beta_1})(\Lambda_N-\lambda_{j(2),\beta(2)})},$$

$$C_3(\Lambda_N) = \sum_{\substack{(j_1,\beta_1)\in Q(\rho^\alpha, 9r_1)\\ (j_2,\beta_2)\in Q(\rho^\alpha, 9r_2),\\ (j_3,\beta_3)\in Q(\rho^\alpha, 9r_3)}} \frac{\left(\prod_{k=1,2,3} A(j(k-1),\beta(k-1),j(k),\beta(k))\right)b(N,j(3),\beta(3))}{(\Lambda_N-\lambda_{j(1),\beta(1)})(\Lambda_N-\lambda_{j(2),\beta(2)})}, \tag{43}$$

and $(j(3),\beta(3)) \neq (j,\beta)$. By (36) and lemma 1, $b(N, j(3), \beta(3)) = O(\rho^{-a})$ for $(j(3),\beta(3)) \neq (j,\beta)$. Using this, (34), and taking into account that

$$|\Lambda_N(t) - \lambda_{j(1),\beta(1)}| > \tfrac{1}{3}\rho^a, \qquad |\Lambda_N(t) - \lambda_{j(2),\beta(2)}| > \tfrac{1}{3}\rho^a$$

for $\beta(1) \neq \beta$, $\beta(2) \neq \beta$ (see (37)), we obtain $C_3(\Lambda_N) = O(\rho^{-3a})$. The estimations of the terms of the first, second and third groups imply that the formula (40) can be written in the form

$$(\Lambda_N - \lambda_{j,\beta})b(N, j, \beta) = (C_1(\Lambda_N) + C_2(\Lambda_N))b(N, j, \beta) + O(\rho^{-3a+2\alpha_1}\ln\rho), \tag{44}$$

where $N = N(j, \beta, v, \tau)$, $\Lambda_{N(j,\beta,v,\tau)} = \Lambda_{j,\beta}(v, \tau)$ (see remark 1). Therefore, dividing both parts of (44) by $b(N, j, \beta)$, where $b(N, j, \beta) = 1 + o(1)$ (see (25)), we get

$$\Lambda_{j,\beta} = \lambda_{j,\beta} + C_1(\Lambda_{j,\beta})) + C_2(\Lambda_{j,\beta})) + O(\rho^{-3a+2\alpha_1}\ln\rho). \tag{45}$$

The calculations in appendices C and B show that

$$C_1(\Lambda_{j,\beta}(v,\tau)) = \frac{1}{4}\int_F |f_{\delta,\beta+\tau}(x)|^2|\varphi_{j,v}(\langle\delta,x\rangle)|^2\,\mathrm{d}x + O(\rho^{-3a+2\alpha_1}), \tag{46}$$

$$C_2(\Lambda_{j,\beta}(v,\tau)) = O(\rho^{-3a+2\alpha_1}). \tag{47}$$

Therefore, (13) follows from (45). □

Theorem 3. *If (23) and (29) hold, then the eigenfunction $\Psi_{j,\beta}(x)$ corresponding to the eigenvalue $\Lambda_{j,\beta}(v,\tau)$, where $\Lambda_{j,\beta}(v,\tau)$ is defined in theorem 1, satisfies (14).*

Proof. To prove (14), we need to show that

$$\sum_{(j',\beta'):(j',\beta')\neq(j,\beta)} |b(N(j,\beta), j', \beta')|^2 = O(\rho^{-2a}). \tag{48}$$

In [10] (see (6.36) of [10]), we proved that

$$\sum_{(j',\beta')\in S^c(k-1)} |b(N, j', \beta')|^2 = O(\rho^{-2k\alpha_2}(\ln\rho)^2), \tag{49}$$

where $S^c(n) = K_0\backslash S(n)$, $K_0 = \{(j',\beta') : j' \in \mathbb{Z}, \beta' \in \Gamma_\delta, (j',\beta') \neq (j,\beta)\}$, $S(n) = \{(j',\beta') \in K_0 : |\beta - \beta'| \leqslant n\rho^\alpha, |j'\delta| < 10^n h\}$, $h = O\left(\rho^{\frac{1}{2}\alpha_2}\right)$ and k can be chosen such that $k\alpha_2 > a$, $k < p$. Therefore, it is enough to prove that

$$\sum_{(j',\beta')\in S(k-1)} |b(N, j', \beta')|^2 = O(\rho^{-2a}). \tag{50}$$

Using (37), (38), the definition of $S(k-1)$ and the Bessel inequality for the basis

$\{\Phi_{j',\beta'}(x) : j' \in \mathbb{Z}, \beta' \in \Gamma_\delta\}$, we have

$$\sum_{(j',\beta'):(j',\beta')\in S(k-1),\beta'\neq\beta} |b(N,j',\beta')|^2 = \sum_{(j',\beta')} \frac{|(\Psi_N(q-q^\delta),\Phi_{j',\beta'})|^2}{|\Lambda_N-\lambda_{j',\beta'}|^2} = O(\rho^{-2a}). \tag{51}$$

In the case $\beta' = \beta, j' \neq j$, using lemma 1 (we can use it since $|j'\delta| = O(\rho^{\frac{1}{2}\alpha_2})$ for $(j',\beta') \in S(k-1)$), we obtain

$$\sum_{(j',\beta)\in S(k-1),j'\neq j} |b(N,j',\beta)|^2 = O(\rho^{-4a+2\alpha_2}(\ln\rho)^2)K, \tag{52}$$

where K is the number of j' satisfying $(j',\beta) \in S(k-1)$. It is clear that $K = O(\rho^{\frac{1}{2}\alpha_2})$. Since $\alpha_2 < \frac{a}{2}$ (see (11), (13)), the right-hand side of (52) is $O(\rho^{-2a})$. Thus (52), (51) give (50). □

Now we estimate the derivative of $\Lambda_N(t)$ by using the following lemma.

Lemma 2. *Let $\Lambda_N(\beta+\tau+v\delta)$, be a simple eigenvalue of L_t satisfying*

$$|\Lambda_N(\beta+\tau+v\delta) - |\beta+\tau|^2| < |\delta|^{-2}\rho^{\alpha_1} \tag{53}$$

where α_1 is defined in (11), β satisfies (23) and $\beta+\tau+v\delta-t \in \Gamma$. Then

$$|\beta+\tau|\frac{\partial\Lambda_N(t)}{\partial h} = \sum_{j'\in\mathbb{Z},\beta'\in\Gamma_\delta} \langle\beta+\tau,\beta'+\tau\rangle|b(N,j',\beta')|^2, \tag{54}$$

where $\frac{\partial\Lambda_N(t)}{\partial h}$ is the derivative of $\Lambda_N(t)$ in the direction of $h = \frac{\beta+\tau}{|\beta+\tau|}$. Moreover,

$$|b(N,j',\beta')| \leqslant \frac{c_3}{(|\beta'+\tau|^2+|(j'+v)\delta|^2)|\beta'+\tau|^{2d+6}} \tag{55}$$

for all β' satisfying $|\beta'+\tau| \geqslant 4\rho$ and for all $j' \in \mathbb{Z}$.

Theorem 4. *If (23) and (29) hold, then the eigenvalue $\Lambda_{j,\beta}(v,\tau)$, defined in theorem 1, satisfies (15).*

Proof. It follows from (55), (48), (14) that

$$\sum_{j'\in\mathbb{Z},|\beta'+\tau|\geqslant 4\rho} \langle\beta+\tau,\beta'+\tau\rangle|b(N,j',\beta')|^2 = O(\rho^{2-2a}),$$

$$\sum_{j'\in\mathbb{Z},|\beta'+\tau|<4\rho,(j',\beta')\neq(j,\beta)} \langle\beta+\tau,\beta'+\tau\rangle|b(N,j',\beta')|^2 = O(\rho^{2-2a}),$$

$$\langle\beta+\tau,\beta+\tau\rangle|b(N,j,\beta)|^2 = |\beta+\tau|^2 + O(\rho^{2-2a}),$$

where $N = N(j,\beta,v,\tau)$, $\Lambda_{N(j,\beta,v,\tau)} = \Lambda_{j,\beta}(v,\tau)$ (see remark 1). Therefore (15) follows from (54). □

To prove the main results of this paper we need the following lemmas.

Lemma 3. *If $\Lambda_N(\beta+\tau+v\delta)$ is a simple eigenvalue of $L_t(q)$ satisfying $|\Lambda_N(\beta+\tau+v\delta) - |\beta+\tau|^2| < 2\rho^\alpha$, $N \neq N(j,\beta,v,\tau)$, where $\beta+\tau+v\delta-t \in \Gamma$, α is defined in (11), j,β,v,τ satisfy (23), (29), then*

$$|\beta+\tau|\frac{\partial\Lambda_N(t)}{\partial h} < |\beta+\tau|^2 - \frac{1}{4}\rho^{2\alpha_d}.$$

The proof of this lemma is given in section 3. Here we note some reasons of the proof. It follows from (48) that

$$|b(N,j,\beta)|^2 = 1 + O(\rho^{-2a}) \qquad \text{for} \quad N = N(j,\beta). \tag{56}$$

Since $\|\Phi_{j,\beta}(x)\| = 1$, using the Parseval's equality for the orthonormal basis

$\{\Psi_N(x) : N = 1, 2, \ldots, \}$ and (56), we get

$$|b(N, j, \beta)|^2 = O(\rho^{-2a}), \qquad \forall N \neq N(j, \beta). \tag{57}$$

This with the long estimations of the other terms of the series of the right-hand side of (54) implies the proof of lemma 3.

Lemma 4. *Let b be a visible element of Γ_δ and $v \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$. Then there exists $\rho(v)$ such that if $\rho \geqslant \rho(v)$, then there exists $\beta \in S_2(\rho)$ satisfying (29), the relation $v \notin A(\beta, \rho)$ and the inequalities*

$$\frac{1}{3}|\rho|^a < |\langle \beta + \tau, b\rangle| < 3|\rho|^a, \tag{58}$$

$$|\langle \beta + \tau, \gamma\rangle| > \frac{1}{3}|\rho|^a, \qquad \forall \gamma \in S(\delta, b)\backslash\delta\mathbb{R}, \tag{59}$$

$$|\langle \beta + \tau, \gamma\rangle| > \frac{1}{3}|\rho|^{a+2\alpha}, \qquad \forall \gamma \notin S(\delta, b), \qquad |\gamma| < |\rho|^\alpha, \tag{60}$$

$$\int_F |f_{\delta,\beta+\tau}(x)|^2 |\varphi_{n,v}(\langle \delta, x\rangle)|^2 \,\mathrm{d}x < c_4 \rho^{-2a} \tag{61}$$

for $\tau \in F_\delta$, where S_2, $A(\beta, \rho)$, $f_{\delta,\beta+\tau}$, $S(\delta, b)$ are defined in (26), (27), (13), (5).

Theorem 5. *Suppose $q \in W_2^s(F)$, where $W_2^s(F)$ is defined by (1), $s \geqslant 6(3^d(d+1)^2) + d$, and the band functions are known. Then the spectral invariants $\mu_j(v)$ for $j \in \mathbb{Z}$, $v \in [0, 1)$ and (4), (7), (16), (17), (20) can be determined constructively.*

Proof. Let $j \in \mathbb{Z}$ and $v \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$. In [10] (see lemma 3.7 of [10]) we proved that $\left(\varepsilon(\rho), \frac{1}{2} - \varepsilon(\rho)\right) \cup \left(\frac{1}{2} + \varepsilon(\rho), 1 - \varepsilon(\rho)\right) \subset W(\rho)$, where $W(\rho)$ is defined in (27) and $\varepsilon(\rho) \to 0$ as $\rho \to \infty$. Therefore $v \in W(\rho)$ for $\rho \gg 1$. On the other hand, by lemma 4, there exists $\beta \in S_2(\rho)$ such that (29), the relation $v \notin A(\beta, \rho)$ and (58)–(61) hold. Then $v \in S_3(\beta, \rho)$ (see (27)). Thus j, β, v satisfy (23) and β satisfies (29), (58)–(61) for $\rho \gg 1$. Replacing ρ by $\rho_k \equiv 3^k \rho$ for $k = 1, 2, \ldots,$ in the same way, we obtain the sequence $\beta_1, \beta_2, \ldots,$ such that $\beta_k \in S_2(\rho_k)$, $v \in S_3(\beta_k, \rho_k)$ and the relations obtained from (29), (58)–(61) by replacing β, ρ with β_k, ρ_k hold. Now take τ from F_δ and consider the band functions $\Lambda_N(\beta_k + \tau + v\delta)$ for $N = 1, 2, \ldots.$ Let $A_k(v)$ be the set of all $\tau \in F_\delta$ for which there exists N satisfying the conditions:

$$|\Lambda_N(\beta_k + \tau + v\delta) - |\beta_k + \tau|^2| < (\rho_k)^{\frac{\alpha}{2}}, \tag{62}$$

$$\Lambda_N(\beta_k + \tau + v\delta) \text{ is a simple eigenvalue,} \tag{63}$$

$$\left||\beta_k + \tau| \frac{\partial \Lambda_N(\beta_k + \tau + v\delta)}{\partial h} - |\beta_k + \tau|^2\right| < \rho_k^{2-2a+\alpha}, \tag{64}$$

where $h = \frac{\beta_k + \tau}{|\beta_k + \tau|}$. By (12), (39) and theorem 4, $\Lambda_{j',\beta_k}(v, \tau)$ for $|j'| < \rho_k^{\frac{\alpha}{5}}$ and for

$$\beta_k \in S_2(\rho_k), \qquad \beta_k \notin \bigcup_{b \in \Gamma_\delta(p\rho_k^\alpha)} V_b^\delta(\rho_k^a), \qquad v \in S_3(\beta_k, \rho_k), \qquad \tau \in S_4(\beta_k, j', v, \rho_k) \tag{65}$$

satisfy the conditions (62)–(64). Therefore $S_4(\beta_k, j', v, \rho_k) \subset A_k(v)$ for $|j'| < \rho_k^{\frac{\alpha}{5}}$ and hence $A_k(v)$ is not an empty set. Moreover , it follows from (63) that $\Lambda_N(\beta_k + \tau + v\delta)$ and $\frac{\partial \Lambda_N(\beta_k + \tau + v\delta)}{\partial h}$ are measurable functions of τ and hence $A_k(v)$ is a measurable set. Let

$$\Lambda_{N_1}(\beta_k + \tau + v\delta) < \Lambda_{N_2}(\beta_k + \tau + v\delta) < \cdots < \Lambda_{N_{n(k)}}(\beta_k + \tau + v\delta) \tag{66}$$

be the eigenvalues of L_t satisfying (62)–(64). Using theorem 4 and lemma 3, we see that if (65) holds for $j' \in S_1(\rho_k)$, then there exist $(j_1, \beta_k), (j_2, \beta_k), \ldots, (j_{n(k)}, \beta_k)$ such that

$N_i = N(j_i, \beta_k)$ for $i = 1, 2, \ldots, n(k)$, i.e.,

$$\Lambda_{N_i}(\beta_k + \tau + v\delta) = \Lambda_{j_i,\beta_k}(v, \tau) \qquad \text{for} \quad i = 1, 2, \ldots, n(k) \tag{67}$$

(see remark 1). Let $\mu_j(v)$ be $i(j)$th eigenvalue of the operator T_v if the eigenvalues of T_v are numbered in the increasing order. (Note that the eigenvalues of the operator T_v for $v \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$ are simple (see [6])). Using (66), (67) and (12), (13) we obtain that if k is a large number and (65) holds for all j' such that $\mu_{j'} \leqslant \mu_j$, then

$$\Lambda_{N_{i(j)}}(\beta_k + \tau + v\delta) = |\beta_k + \tau|^2 + \mu_j(v) + O\left(\rho_k^{-a}\right), \tag{68}$$

$$\Lambda_{N_{i(j)}}(\beta_k + \tau + v\delta) = |\beta_k + \tau|^2 + \mu_j(v) + \frac{1}{4}\int_F \left|f_{\delta,\beta_k+\tau}^2\right| |\varphi_{j,v}|^2 \, \mathrm{d}x + O\left(\rho_k^{-3a+2\alpha_1} \ln \rho_k\right). \tag{69}$$

For $\tau \in A_k(v)$, take $i(j)$th element $\Lambda_{N_{i(j)}}(\beta_k + \tau + v\delta)$ (see (66)) of the set of the eigenvalues satisfying (62)–(64) and consider the integral

$$J(A_k) = \frac{1}{\mu(F_\delta)} \int_{A_k(v)} (\Lambda_{N_{i(j)}}(\beta_k + \tau + v\delta) - |\beta_k + \tau|^2) \, \mathrm{d}\tau.$$

This integral is a sum of $J(S_4')$ and $J(A_k(v)\backslash S_4')$, where S_4' denotes the intersection of $S_4(\beta_k, j', v, \rho_k)$ for all j' such that $\mu_{j'} \leqslant \mu_j$. If $\tau \in S_4'$ and k is a large number, then (68) holds. Thus, using (68) and (28) for $\rho = \rho_k$, we get $J(S_4') = \mu_j(v) + O(\rho_k^{-\alpha})$. On the other hand, the inclusion $A_k(v) \subset F_\delta$, (28), and (62) imply that $\mu(A_k(v)\backslash S_4') = O\left(\rho_k^{-\alpha}\right)$ and $J\left(A_k(v)\backslash S_4'\right) = O\left(\rho_k^{-\frac{\alpha}{2}}\right)$. These equalities yield $J(A_k(v)) = \mu_j(v) + O(\rho_k^{-\frac{\alpha}{2}})$. Letting $k \to \infty$, we find $\mu_j(v)$ for $j \in \mathbb{Z}$ and $v \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$. Since $\mu_j(0)$ and $\mu_j\left(\frac{1}{2}\right)$ are the end points of the interval $\left\{\mu_j(v) : v \in \left(0, \frac{1}{2}\right)\right\}$, the invariant $\mu_j(v)$ is determined constructively for all $v \in [0, 1)$. In appendix D, we constructively determine (17) from the asymptotic formulae for $\mu_j(v)$.

Now using (69) and taking into account that the invariants $\mu_j(v)$ are determined, we determine the invariant (4) as follows. Let $B(\beta_k, v)$ be the set of $\tau \in F_\delta$ for which there exists N satisfying (63), (64) and

$$|\Lambda_N(\beta_k + \tau + v\delta) - |\beta_k + \tau|^2 - \mu_j(v)| < \rho_k^{-2a+\frac{\alpha}{2}}. \tag{70}$$

For $\tau \in B(\beta_k, v)$, take one of the eigenvalues $\Lambda_N(\beta_k + \tau + v\delta)$ satisfying (63), (64), (70) and consider

$$J'(B(\beta_k, v)) = \frac{|\langle \beta_k + \tau, b\rangle|^2}{\mu(F_\delta)|b|^4} \int_{B(\beta_k,v)} (\Lambda_N(\beta_k + \tau + v\delta) - |\beta_k + \tau|^2 - \mu_j(v)) \, \mathrm{d}\tau.$$

This integral is a sum of $J'(S_4)$ and $J'(B(\beta_k, v)\backslash S_4)$. If $\tau \in S_4$ and k is a large number, then arguing as above and taking into account that $\mu_j(v)$ is a simple eigenvalue, we see that only the eigenvalue $\Lambda_{N_{i(j)}}(\beta_k + \tau + v\delta)$ (see (69)) satisfies (63), (64), (70). Hence, in $J'(S_4)$ instead of $\Lambda_N(\beta_k + \tau + v\delta)$ we must take$\Lambda_{N_{i(j)}}(\beta_k + \tau + v\delta)$. Therefore, using (69), we get

$$J'(S_4) = \frac{|\langle \beta_k + \tau, b\rangle|^2}{4\mu(F_\delta)|b|^4} \int_{S_4} \int_F \left|f_{\delta,\beta_k+\tau}(x)\varphi_{j,v}(\langle \delta, x\rangle)\right|^2 \mathrm{d}x \, \mathrm{d}\tau + O\left(\rho_k^{2\alpha_1 - a} \ln \rho\right). \tag{71}$$

Moreover, using (70), (58) and $\mu(B(\beta_k, v)\backslash S_4) = O\left(\rho_k^{-\alpha}\right)$ (see (28)), we obtain

$$J'(B(\beta_k, v)\backslash S_4) = O\left(\rho_k^{-\frac{\alpha}{2}}\right). \tag{72}$$

Substituting the decomposition $|\delta|^{-2}\langle\gamma,\delta\rangle\delta + |b|^{-2}\langle\gamma,b\rangle b$ of γ for $\gamma \in S(\delta,b)$, $|\gamma| < |\rho_k|^\alpha$ into the denominator of the fraction in $f_{\delta,\beta_k+\tau}(x)$ (for definition of this function see (13)) and using (58), (60), we have

$$\lim_{k\to\infty} |b|^{-2}\langle\beta_k+\tau,b\rangle f_{\delta,\beta_k+\tau}(x) = \sum_{\gamma\in S(\delta,b)\backslash\delta\mathbb{R}} \frac{\gamma}{\langle\gamma,b\rangle} q_\gamma\, \mathrm{e}^{\langle\gamma,x\rangle} \equiv q_{\delta,b}(x), \qquad (73)$$

where $q_{\delta,b}(x)$ is defined in (5) and the convergence of the series (5) is proved in the proof of lemma 4. This, with (71) and (72), implies that

$$\lim_{k\to\infty} J'(B(\beta_k,v) = \int_F |q_{\delta,b}(x)|^2 |\varphi_{j,v}(\langle\delta,x\rangle)|^2\,\mathrm{d}x \equiv J(\delta,b,j,v) \qquad (74)$$

(see (4)). In (74), letting $j\to\infty$ and using (6), we get the invariant $J_0(\delta,b)$ (see (7)). Then, we find the other invariants $J_1(\delta,b)$, $J_2(\delta,b),\ldots,$ of (7) as follows

$$J_1 = \lim_{j\to\infty}(J-J_0)j, \qquad J_2 = \lim_{j\to\infty}((J-J_0)j^2 - J_1 j),\ldots.$$

In appendix D, using the asymptotic formulae for the eigenfunctions of $T_v(Q)$, we constructively determine the invariants (16), (20) from (7) and (17) □

3. The proofs of the lemmas

The proof of lemma 1. To prove this lemma, we use the following formula obtained from (40) by replacing j and r_1 with j' and r, respectively,

$$(\Lambda_{N(j,\beta)} - \lambda_{j',\beta})b(N,j',\beta) = O(\rho^{-p\alpha}) + \sum_{\substack{(j_1,\beta_1)\in Q(\rho^\alpha,9r)\\(j_2,\beta_2)\in Q(\rho^\alpha,90r)}} \frac{A(j,\beta,j'(1),\beta(1))A(j'(1),\beta(1),j'(2),\beta(2))b(N,j'(2),\beta(2))}{\Lambda_N - \lambda_{j'+j_1,\beta+\beta_1}}, \qquad (75)$$

where $j'(k) = j' + j_1 + j_2 + \cdots + j_k$ for $k = 0,1,2,\ldots$. By (36), we have

$$b(N,j'(2),\beta(2)) = O(\rho^{-a}) \qquad (76)$$

for $\beta(2)\neq\beta$. If $j'(2)\neq j$, then using (12) and taking into account that $v\in S_3(\beta,\rho)\subset W(\rho)$ (see the definition of $W(\rho)$ in (27)), we obtain

$$|\Lambda_{N(j,\beta)} - \lambda_{j',\beta}| > \frac{1}{\ln\rho}. \qquad (77)$$

Therefore, using (34), remark 2 and (36), we see that $b(N,j'(2),\beta) = O(\rho^{-a}\ln\rho)$ for $j'(2)\neq j$. Using this, (34), and the estimations (37), (76), we see that the sum of the terms of the right-hand side of (75) with multiplicand $b(N,j'(2),\beta(2))$ for $(j'(2),\beta(2))\neq(j,\beta)$ is $O(\rho^{-2a}\ln\rho)$. It means that formula (75) can be written in the form

$$(\Lambda_N - \lambda_{j',\beta})b(N,j',\beta) = O(\rho^{-2a}\ln\rho) + C_1(j',\Lambda_N)b(N,j,\beta), \qquad (78)$$

where

$$C_1(j',\Lambda_N) = \sum_{(j_1,\beta_1)\in Q(\rho^\alpha,9r)} \frac{A(j',\beta,j'+j_1,\beta+\beta_1)A(j'+j_1,\beta+\beta_1,j,\beta)}{\Lambda_N - \lambda_{j'+j_1,\beta+\beta_1}}. \qquad (79)$$

By (12), (37), (34), we have $\frac{1}{\Lambda_N - \lambda_{j'+j_1,\beta+\beta_1}} = \frac{1}{\lambda_{j,\beta} - \lambda_{j'+j_1,\beta+\beta_1}} = O(\rho^{-3a})$,

$$C_1(j',\Lambda_N) = C_1(j',\lambda_{j,\beta}) + O(\rho^{-3a}), \qquad (80)$$

where $C_1(j', \lambda_{j,\beta})$ is obtained from $C_1(j', \Lambda_N)$ by replacing Λ_N with $\lambda_{j,\beta}$ in the denominator of the fractions in (79). In appendix A, we prove that

$$C_1(j', \lambda_{j,\beta}) = O(\rho^{-2a} r^2) \tag{81}$$

for $|j'\delta| < r$, $(j_1, \beta_1) \in Q(\rho^\alpha, 9r)$, $j \in S_1$. Therefore, dividing both sides of (78) by $\Lambda_N - \lambda_{j',\beta}$ and using (77), (80), (81), we get the proof of the lemma. □

The proof of lemma 2. We find the derivative of $\Lambda_N(t)$ by using

$$\frac{\partial \Lambda_N(t)}{\partial t_j} = 2t_j - 2\mathrm{i}\left(\frac{\partial}{\partial x_j}\Phi_{N,t}, \Phi_{N,t}\right),$$

where $\Phi_{N,t}(x) = \mathrm{e}^{-\mathrm{i}\langle t,x\rangle}\Psi_{N,t}(x)$, $t = (t_1, t_2, \ldots, t_d)$ (see (5.12) of [10]). Then,

$$\frac{\partial \Lambda_N(t)}{\partial h} = \sum_{j=1}^{d} h_j \frac{\partial \Lambda_N(t)}{\partial t_j} = 2\langle h, t\rangle - 2\mathrm{i}\left(\frac{\partial}{\partial h}\Phi_{N,t}, \Phi_{N,t}\right). \tag{82}$$

To compute $\frac{\partial}{\partial h}\Phi_{N,t}(x)$, we prove that the decomposition

$$\Phi_{N,t}(x) = \sum_{j'\in\mathbb{Z},\beta'\in\Gamma_\delta} b(N, j', \beta')\,\mathrm{e}^{\mathrm{i}\langle\beta'+\tau-t,x\rangle}\varphi_{j'}(\langle\delta, x\rangle) \tag{83}$$

of $\Psi_{N,t}$ by basis $\{\Psi_{j,\beta} : j \in \mathbb{Z}, \beta \in \Gamma_\delta\}$ can be differentiated term by term. Since $\langle\delta, h\rangle = 0$,

$$\frac{\partial}{\partial h}\,\mathrm{e}^{\mathrm{i}\langle\beta'+\tau-t,x\rangle}\varphi_{j'}(\langle\delta, x\rangle) = \mathrm{i}\langle\beta' + \tau - t, h\rangle\,\mathrm{e}^{\mathrm{i}\langle\beta'+\tau-t,x\rangle}\varphi_{j'}(\langle\delta, x\rangle),$$

we need to prove that

$$\frac{\partial}{\partial h}\Phi_{N,t}(x) = \sum_{j'\in\mathbb{Z},\beta'\in\Gamma_\delta} \mathrm{i}\langle\beta' + \tau - t, h\rangle b(N, j', \beta')\,\mathrm{e}^{\mathrm{i}\langle\beta'+\tau-t,x\rangle}\varphi_{j'}(\langle\delta, x\rangle). \tag{84}$$

Therefore, we consider the convergence of these series by estimating the multiplicand $b(N, j', \beta')$. First, we estimate this multiplicand for $(j', \beta') \in E$, where $E = \{(j', \beta') : |(j' + v)\delta|^2 + |\beta' + \tau|^2 \geqslant 9\rho^2\}$, by using the formula

$$b(N, j', \beta') = \frac{(\Psi_{N,t}, (q - q^\delta)\Phi_{j',\beta'})}{\Lambda_N - \lambda_{j',\beta'}}, \tag{85}$$

which can be obtained from (38) by replacing $j + j_1, \beta + \beta_1$ with j', β'. By (23), (53)

$$|\Lambda_N| < 3\rho^2. \tag{86}$$

This inequality, the condition $(j', \beta') \in E$, definition of $\lambda_{j',\beta'}$, and (39) give

$$\lambda_{j',\beta'} - \Lambda_N > \tfrac{1}{2}(|(j' + v)\delta|^2 + |\beta' + \tau|^2) > \rho^2 \tag{87}$$

for $(j', \beta') \in E$. Therefore, (85) implies that

$$|b(N, j', \beta')| \leqslant \frac{c_5}{|(j' + v)\delta|^2 + |\beta' + \tau|^2}, \qquad \forall\,(j', \beta') \in E. \tag{88}$$

Now we obtain the high-order estimation for $b(N, j', \beta')$ when $|\beta' + \tau| \geqslant 4\rho$. In this case, to estimate $b(N, j', \beta')$ we use the iterations of the formula in remark 2. To iterate this formula, we use the relations $|\beta' + \tau - \beta_1 - \beta_2 - \cdots - \beta_k|^2 > \frac{3}{4}|\beta' + \tau|^2$ for $k = 1, 2, \ldots, d + 3$, where $|\beta_i| < \rho^\alpha$ for $i = 0, 1, \ldots, k$. This and (86) give

$$\lambda_{j'(k),\beta'(k)} - \Lambda_N > \tfrac{1}{5}|\beta' + \tau|^2, \qquad \forall\,|\beta' + \tau| \geqslant 4\rho, \tag{89}$$

where $\beta'(k) = \beta' + \beta_1 + \beta_2 + \cdots + \beta_k$. Moreover, if $|j'\delta| < c$, where c is a positive number, then $(j_k, \beta_k) \in Q(\rho^\alpha, 10^{k-1}9c)$. These conditions on j' and j_1 imply that $|j'(1)\delta| < 10c$. Therefore, in the formula in remark 2 replacing j', β', r by $j'(1), \beta'(1), 10c$, we get

$$b(N, j'(1), \beta'(1)) = O(\rho^{-p\alpha}) + \sum_{(j_2,\beta_2)\in Q(\rho^\alpha,90c)} \frac{A(j'(1), \beta'(1), j'(2), \beta'(2))b(N, j'(2), \beta'(2))}{\Lambda_N - \lambda_{j'(1),\beta'(1)}}.$$

In the same way, we obtain

$$b(N, j'(k), \beta'(k)) = O(\rho^{-p\alpha}) + \sum_{(j_{k+1},\beta_{k+1})\in Q(\rho^\alpha,(10^k)9c)} \frac{A(j'(k), \beta'(k), j'(k+1), \beta'(k+1))b(N, j'(k+1), \beta'(k+1))}{\Lambda_N - \lambda_{j'(k),\beta'(k)}} \tag{90}$$

for $k = 1, 2, \ldots$. In the formula in remark 2 for $r = c$, using this formula for $k = 1, 2, \ldots, d+3$ successively, we get

$$b(N, j', \beta') = \sum \left(\prod_{i=0}^{d+3} \frac{A(j'(i), \beta'(i), j'(i+1), \beta'(i+1))}{\Lambda_N - \lambda_{j'(i),\beta'(i)}}\right) b(N, j'(d+4), \beta'(d+4)), \tag{91}$$

where sum is taken under conditions

$(j_1, \beta_1) \in Q(\rho^\alpha, 9c), (j_2, \beta_2) \in Q(\rho^\alpha, 90c), \ldots, (j_{d+4}, \beta_{d+4}) \in Q(\rho^\alpha, (10^{d+3})9c)$. Now using (34), (87) and (89), we obtain the proof of (55). It follows from (88) and (55) that the series in (83) can be term-by-term differentiated and (84) holds. Substituting (84) into (82) and using the Parseval equality, by direct calculation, we obtain the proof of the lemma. □

The proof of lemma 3. By lemma 2, we have

$$|\beta + \tau| \frac{\partial \Lambda_N(t)}{\partial h} = \sum_{j'\in\mathbb{Z},\beta'\in\Gamma_\delta} \langle\beta + \tau, \beta' + \tau\rangle |b(N, j', \beta')|^2 = \sum_{i=1}^{7} C_i, \tag{92}$$

where

$$C_i = \sum_{\beta'\in A_i} \sum_{j'\in\mathbb{Z}} \langle\beta + \tau, \beta' + \tau\rangle |b(N, j', \beta')|^2 \tag{93}$$

and A_i is defined as follows:

$A_1 = \{\beta' \in \Gamma_\delta : \beta' + \tau \notin R_\delta(4\rho)\}$, where $R_\delta(c) = \{x \in H_\delta : |x| < c\}$,

$A_2 = \left\{\beta' \in \Gamma_\delta : \beta' + \tau \in R_\delta(4\rho)\backslash R_\delta\left(H + \frac{1}{9}\rho^{a-1}\right)\right\}$,

$A_3 = \left\{\beta' \in \Gamma_\delta : \beta' + \tau \in R_\delta\left(H + \frac{1}{9}\rho^{a-1}\right)\backslash R_\delta(H + \rho^{\alpha_d-1}), |\beta - \beta'| \geqslant \rho^{a-2\alpha}\right\}$,

$A_4 = \left\{\beta' \in \Gamma_\delta : \beta' + \tau \in R_\delta\left(H + \frac{1}{9}\rho^{a-1}\right)\backslash R_\delta(H + \rho^{\alpha_d-1}), |\beta - \beta'| < \rho^{a-2\alpha}\right\}$,

$A_5 = \left\{\beta' \in \Gamma_\delta : \beta' + \tau \in R_\delta\left(H + \rho^{\alpha_d-1}\right)\backslash R_\delta(H - \rho^{2\alpha_d-1}), |\beta - \beta'| \geqslant \rho^{\alpha_d}\right\}$,

$A_6 = \left\{\beta' \in \Gamma_\delta : \beta' + \tau \in R_\delta\left(H + \rho^{\alpha_d-1}\right)\backslash R_\delta(H - \rho^{2\alpha_d-1}), |\beta - \beta'| < \rho^{\alpha_d}\right\}$,

$A_7 = \left\{\beta' \in \Gamma_\delta : \beta' + \tau \in R_\delta\left(H - \rho^{2\alpha_d-1}\right)\right\}$, where $H = |\beta + \tau|$, $\beta \in S_2(\rho)$, and hence by the definition of $S_2(\rho)$ (see (26)), H satisfies the inequalities

$$\tfrac{1}{2}\rho < H < \tfrac{3}{2}\rho. \tag{94}$$

First, we prove that

$$C_i = O(\rho^{2-2a}), \qquad \forall i = 1, 2, 4, 6. \tag{95}$$

It follows from (55) that (95) holds for $i = 1$. To prove (95) for $i = 2$, we use (85) and show that

$$\lambda_{j',\beta'} - \Lambda_N(t) > c_6\rho^a. \tag{96}$$

First, let us prove (96). By the condition $|\Lambda_N(\beta+\tau+v\delta) - |\beta+\tau|^2| < 2\rho^\alpha$ of the lemma, we have

$$\Lambda_N = H^2 + O(\rho^\alpha). \tag{97}$$

If $\beta' \in A_2$, then using (94), definition of $\lambda_{j',\beta'}$ and (39), we have

$$\lambda_{j',\beta'} > H^2 + c_7\rho^a. \tag{98}$$

This, (97), and the inequality $a > \alpha$ imply (96). Now using (96), (85), the inequalities $|\beta+\tau| < \frac{3}{2}\rho$, $|\beta'+\tau| < 4\rho$ and the Bessel inequality, we get the proof of (95) for $i = 2$.

To prove (95) for $i = 4$, we use the inequality $C_4 < c_8\rho^2(C_{4,1} + C_{4,2})$, where

$$C_{4,1} = \sum_{\beta'\in A_4} \sum_{j':|j'\delta|\geqslant \frac{1}{30}\rho^{\frac{a}{2}}} |b(N, j', \beta')|^2,$$

$$C_{4,2} = \sum_{\beta'\in A_4} \sum_{j':|j'\delta|< \frac{1}{30}\rho^{\frac{a}{2}}} |b(N, j', \beta')|^2,$$

and prove that

$$C_{4,i} = O(\rho^{-2a}), \qquad \forall i = 1, 2. \tag{99}$$

It is clear that if $\beta' \in A_4$ and $|j'\delta| \geqslant \frac{1}{30}\rho^{\frac{a}{2}}$, then (98) holds. Therefore, repeating the proof of (95) for $i = 2$, we get the proof of (99) for $i = 1$.

Now we prove (99) for $i = 2$. It follows from (85) that

$$C_{4,2} = \sum_{\beta'\in A_4} \sum_{j':|j'\delta|< \frac{1}{30}\rho^{\frac{a}{2}}} \frac{|(\Psi_N, (q - q^\delta)\Phi_{j',\beta'})|^2}{|\Lambda_N(t) - \lambda_{j',\beta'}|^2}. \tag{100}$$

Since $\alpha_d > \alpha$, it follows from (97) that the inequality $\lambda_{j',\beta'} - \Lambda_N(t) > c_9\rho^{\alpha_d}$ holds for $\beta' \in A_4$, $|j'\delta| < \rho^{\frac{a}{2}}$. Therefore, using (39), we obtain

$$\sum_{j':|j'\delta|< \frac{1}{30}\rho^{\frac{a}{2}}} \frac{1}{|\Lambda_N(t) - \lambda_{j',\beta'}|^2} < c_{10}, \qquad \forall \beta' \in A_4, \tag{101}$$

where c_{10} does not depend on β'. Using this in (100) and denoting

$$|(\Psi_N, (q - q^\delta)\Phi_{n(\beta'),\beta'})| = \max_{j':|j'\delta|< \frac{1}{30}\rho^{\frac{a}{2}}} |(\Psi_N, (q - q^\delta)\Phi_{j',\beta'})|$$

(if max is obtained for several index $n(\beta')$, then we take one of them), we get

$$C_{4,2} < c_{11} \sum_{\beta'\in A_4} |(\Psi_N, (q - q^\delta)\Phi_{n(\beta'),\beta'})|^2.$$

Now using (33), (34) and then (85), we obtain

$$C_{4,2} < c_{12}\rho^{-p\alpha} + c_{12} \sum_{\beta'\in A_4} |b(N, n(\beta') + j_1(\beta'), \beta' + \beta_1(\beta'))|^2$$

$$= c_{12}\rho^{-p\alpha} + c_{12} \sum_{\beta'\in A_4} \frac{|(\Psi_N, (q - q^\delta)\Phi_{n(\beta')+j_1(\beta'),\beta'+\beta_1(\beta')})|^2}{|\Lambda_N - \lambda_{n(\beta')+j_1(\beta'),\beta'+\beta_1(\beta')}|^2}, \tag{102}$$

where

$$|b(N, n(\beta') + j_1(\beta'), \beta' + \beta_1(\beta'))| = \max_{(j_1,\beta_1)\in Q(\rho^\alpha, 9\frac{1}{30}\rho^{\frac{a}{2}})} |b(N, n(\beta') + j_1, \beta' + \beta_1)|.$$

To estimate $C_{4,2,}$ let us prove that

$$|\Lambda_N - \lambda_{n(\beta')+j_1(\beta'),\beta'+\beta_1(\beta')}| > \tfrac{1}{8}\rho^a. \tag{103}$$

The inclusion $(j_1, \beta_1) \in Q(\rho^\alpha, 9\frac{1}{30}\rho^{\frac{a}{2}})$ and the condition $|j'\delta| < \frac{1}{30}\rho^{\frac{a}{2}}$ imply that $|n(\beta')\delta + j_1(\beta')\delta| < \frac{1}{3}\rho^{\frac{a}{2}}$ and by (39) $|\mu_{n(\beta')+j_1(\beta')}| < \frac{1}{8}\rho^a$. Therefore, by (97), to prove (103) it is enough to show that

$$|H^2 - |\beta' + \beta_1 + \tau|^2| > \tfrac{3}{8}\rho^a, \qquad \forall\, \beta' \in A_4, \qquad \beta_1 \in \Gamma_\delta(p\rho^\alpha). \tag{104}$$

Since $||\beta' + \tau|^2 - H^2| < \frac{1}{2}\rho^a$ (see the definition of A_4 and use (94)), we need to prove that

$$||\beta' + \tau|^2 - |\beta' + \beta_1 + \tau|^2| > \tfrac{7}{8}\rho^a, \qquad \forall\, \beta' \in A_4, \qquad \beta_1 \in \Gamma_\delta(p\rho^\alpha). \tag{105}$$

Using $|\beta - \beta'| < \rho^{a-2\alpha}$ (see definition of A_4), by calculations, we get

$$\begin{aligned}|\beta' + \tau|^2 - |\beta' + \beta_1 + \tau|^2 &= -2\langle \beta' + \tau, \beta_1\rangle - |\beta_1|^2 \\ &= -2\langle \beta + \tau, \beta_1\rangle - |\beta_1|^2 - 2\langle \beta' - \beta, \beta_1\rangle = -(|\beta + \beta_1 + \tau|^2 - |\beta + \tau|^2) + o(\rho^a).\end{aligned}$$

This and (31) imply that (105) and hence (103) holds. Now to estimate the right-hand side of (102) we prove that if $\beta' \in A_4$, $\beta'' \in A_4$ and $\beta' \neq \beta''$, then

$$\beta' + \beta_1(\beta') \neq \beta'' + \beta_1(\beta''). \tag{106}$$

Assume that they are equal. Then, we have $\beta'' = \beta' + b$, where $b \in \Gamma_\delta(2\rho^\alpha)$, since $\beta_1(\beta') \in \Gamma_\delta(\rho^\alpha)$, $\beta_1(\beta'') \in \Gamma_\delta(\rho^\alpha)$. It easily follows from the inclusions $\beta' \in A_4$, $\beta' + b \in A_4$ that $||\beta' + \tau|^2 - |\beta' + \tau + b|^2| < \frac{1}{2}\rho^a$ which contradicts (105). Thus (106) is proved. Therefore, using (102), (103) and the Bessel inequality, we obtain the proof of (99) for $i = 2$. Hence (95) is proved for $i = 4$.

Now we prove (95) for $i = 6$. First, we note that $A_6 = \{\beta\}$. Indeed, if $\beta' \neq \beta$ and $\beta' \in A_6$, then we have $\beta' = \beta + b$, where $b \in \Gamma_\delta(\rho^{\alpha_d})$, and from the relations $\beta \notin V_b^\delta(\rho^{\frac{1}{2}})$ (see (23) and the definition of S_2), $|\beta + \tau| = H$, we obtain that $||\beta' + \tau|^2 - H^2| > \frac{1}{2}\rho^{\frac{1}{2}}$ which contradicts the inclusion $\beta' + \tau \in R_\delta(H + \rho^{\alpha_d - 1})$. Hence,

$$C_6 = \sum_{j'\in\mathbb{Z}} \langle \beta + \tau, \beta + \tau\rangle |b(N, j', \beta)|^2 = H^2 \sum_{j'\in\mathbb{Z}} |b(N, j', \beta)|^2 = H^2 \sum_{i=1}^{3} C_{6,i},$$

where

$$C_{6,1} = |b(N, j, \beta)|^2, \qquad C_{6,2} = \sum_{|j'\delta|\geqslant \frac{1}{30}\rho^{\frac{a}{2}}} |b(N, j', \beta)|^2,$$

$$C_{6,3} = \sum_{|j'\delta|<\frac{1}{30}\rho^{\frac{a}{2}},\, j'\neq j} |b(N, j', \beta)|^2.$$

To prove (95) for $i = 6$, we show that

$$C_{6,i} = O(\rho^{-2a}), \qquad \forall\, i = 1, 2, 3. \tag{107}$$

By (57), this equality holds for $i = 1$. For $|j'\delta| \geqslant \frac{1}{30}\rho^{\frac{a}{2}}$, the inequality (96) holds. Therefore, repeating the proof of (95) for $i = 2$, we get the proof of (107) for $i = 2$. Arguing as in the proof of (99) for $i = 2$, we obtain the proof of (107) for $i = 3$. Thus, (95) is proved for $i = 6$.

Now we prove that

$$C_i \leqslant \sum_{\beta'\in A_i}\sum_{j'\in Z}|b(N,j',\beta')|^2\left(H^2-\frac{1}{3}\rho^{2\alpha_d}\right) \tag{108}$$

for $i=3,5,7$. Consider the triangle generated by vectors $\beta+\tau$, $\beta'+\tau$, $\beta-\beta'$. For $\beta'\in A_3$, we have $H+\rho^{\alpha_d-1}\leqslant|\beta'+\tau|\leqslant H+\frac{1}{9}\rho^{a-1}$, $|\beta-\beta'|\geqslant\rho^{a-2\alpha}$. Let θ be the angle between the vectors $\beta+\tau$, and $\beta'+\tau$. If $|\theta|\leqslant\frac{\pi}{2}$, then using the cosine theorem, we get

$$|\langle\beta+\tau,\beta'+\tau\rangle| = \tfrac{1}{2}(|\beta+\tau|^2+|\beta'+\tau|^2-|\beta-\beta'|^2) < H^2-\tfrac{1}{3}\rho^{2\alpha_d},$$

since $a-2\alpha>\alpha_d$. Using this and taking into account that $\langle\beta+\tau,\beta'+\tau\rangle<0$ for $\frac{\pi}{2}<|\theta|\leqslant\pi$, we get the proof of (108) for $i=3$. If $\beta'\in A_5$, $|\theta|\leqslant\frac{\pi}{2}$, then $|\langle\beta+\tau,\beta'+\tau\rangle|\leqslant H^2-\frac{1}{3}\rho^{2\alpha_d}$ and hence (108) holds for $i=5$. If $\beta'\in A_7$, then $|\beta'+\tau|\leqslant H-\rho^{2\alpha_d-1}$ and by (94) we have $|\langle\beta+\tau,\beta'+\tau\rangle|\leqslant H^2-\frac{1}{3}\rho^{2\alpha_d}$, i.e, (108) holds for $i=7$ too. Now (108) and the Bessel inequality imply that

$$C_3+C_5+C_7\leqslant H^2-\tfrac{1}{3}\rho^{2\alpha_d}=|\beta+\tau|^2-\tfrac{1}{3}\rho^{2\alpha_d}.$$

This, (95) and (54) give the proof of lemma 3, since $2-2a<2\alpha_d$ (see (13)). □

The proof of lemma 4. Let n_1 be a positive integer satisfying the inequality $|(n_1+v)\delta|^2\leqslant 4\rho^{1+\alpha_d}<|(n_1+1+v)\delta|^2$. Introduce the following sets

$$D_{b',j}(\rho,v,4)=\{x\in H_\delta : |2\langle x,b'\rangle+|b'|^2+|(j+v)\delta|^2|<4d_\delta\rho^{\alpha_d}\},$$

$$D(\rho,v,4)=\bigcup_{j=-n_1-3}^{n_1}\ \bigcup_{b'\in\Gamma_\delta(\rho^{\alpha_d})} D_{b',j}(\rho,v,4), \tag{109}$$

$$S_2'(\rho,b,v)=\left(\left(V_b^\delta(4\rho^a)\backslash V_b^\delta(\rho^a)\right)\backslash\left(D(\rho,v,4)\cup D_1\left(\rho^{\frac{1}{2}}\right)\cup D_2(\rho^{a+2\alpha})\right)\right)\cap D_3, \tag{110}$$

where

$$D_1\left(\rho^{\frac{1}{2}}\right)=\bigcup_{b'\in\Gamma_\delta\left(\rho^{\alpha_d}\right)} V_{b'}^\delta\left(\rho^{\frac{1}{2}}\right), \qquad D_2(\rho^{a+2\alpha})=\bigcup_{b'\in\Gamma_\delta(p\rho^\alpha)\backslash b\mathbb{R}} V_{b'}^\delta(\rho^{a+2\alpha}),$$

$$D_3=\left(R\left(\tfrac{3}{2}\rho-d_\delta-1\right)\backslash R\left(\tfrac{1}{2}\rho+d_\delta+1\right)\right).$$

Now we prove that the set $S_2'(\rho,b,v)$ contains an element $\beta\in\Gamma_\delta$ satisfying all assertions of lemma 4. First, let us prove that $S_2'(\rho,b,v)\cap\Gamma_\delta$ is a nonempty subset of $S_2(\rho)$, i.e.,

$$S_2'(\rho,b,v)\cap\Gamma_\delta\subset S_2(\rho), \qquad S_2'(\rho,b,v)\cap\Gamma_\delta\neq\emptyset. \tag{111}$$

It follows from the definitions of $S_2'(\rho,b,v)$ and $S_2(\rho)$ (see (23)) that the first relation of (111) holds. To prove the second relation, we consider the set

$$D'(\rho)=\left(V_b^\delta(3\rho^a)\backslash V_b^\delta(2\rho^a)\right)\backslash\left(D(\rho,v,6)\cup D_1\left(2\rho^{\frac{1}{2}}\right)\cup D_2(2\rho^{a+2\alpha})\right)\cap D_4,$$

where $D_4=R\left(\frac{3}{2}\rho-1\right)\backslash R\left(\frac{1}{2}\rho+1\right)$. If $\beta+\tau\in D'(\rho)$, where $\beta\in\Gamma_\delta,\tau\in F_\delta$, then $\beta\in S_2'(\rho,b,v)$. Therefore, $\{\beta+F_\delta:\beta\in S_2'(\rho,b,v)\cap\Gamma_\delta\}$ is a cover of $D'(\rho)$. Hence,

$$|S_2'(\rho,b,v)\cap\Gamma_\delta|\geqslant(\mu(F_\delta))^{-1}\mu(D'(\rho)), \tag{112}$$

where $|S_2'(\rho,b,v)\cap\Gamma_\delta|$ is the number of elements of $S_2'(\rho,b,v)\cap\Gamma_\delta$. Thus, to prove the second relation of (111) we need to estimate $\mu(D'(\rho))$. It is not hard to verify that (see remark 2.1 of [10])

$$\mu\left(\left(V_b^\delta(3\rho^a)\backslash V_b^\delta(2\rho^a)\right)\cap D_4\right)>c_{13}\rho^{d-2+a}. \tag{113}$$

Now we estimate $\mu\big(\big(V_b^\delta(3\rho^a)\big\backslash V_b^\delta(2\rho^a)\big)\cap D_1\big(2\rho^{\frac12}\big)\cap D_4\big)$. If $b'\in(b\mathbb{R})\cap\Gamma_\delta(\rho^{\alpha_d})$, then one can easily verify that $V_{b'}^\delta\big(2\rho^{\frac12}\big)\cap D_4\subset V_b^\delta(2\rho^a)\cap D_4$. Therefore, we need to estimate the measure of $V_b^\delta(3\rho^a)\cap V_{b'}^\delta\big(2\rho^{\frac12}\big)\cap D_4$ for $b'\in\Gamma_\delta(\rho^{\alpha_d})\backslash b\mathbb{R}$. For this, we turn the coordinate axes so that the direction of$(1,0,0,\ldots,0)$ coincides with the direction of b', and the plane generated by b,b' coincides with the plane $\{(x_1,x_2,0,\ldots,0):x_1\in\mathbb{R},x_2\in\mathbb{R}\}$, i.e., $b'=(|b'|,0,0,\ldots,0)$, $b=(b_1,b_2,0,\ldots,0)$. Then the condition $x\in V_b^\delta(3\rho^a)\cap V_{b'}^\delta\big(2\rho^{\frac12}\big)\cap D_4$ implies that

$$x_1|b'|=O\big(\rho^{\frac12}\big),\qquad x_1b_1+x_2b_2=O(\rho^a),\qquad x_1^2+x_2^2+\cdots+x_{d-1}^2=O(\rho^2).\tag{114}$$

First, the equality of (114) shows that $x_1=O\big(\rho^{\frac12}\big)$. Since b' and b are linearly independent vectors of Γ_δ, we have $|b'||b_2|\geqslant\mu(F_\delta)$, where $|b'|<\rho^{\alpha_d}$. Therefore, $|b_2|\geqslant\mu(F_\delta)\rho^{-\alpha_d}$ and the second equality of (114) implies that $x_2=O(\rho^{a+\alpha_d})$. The third equality of (114) implies that $V_b^\delta(3\rho^a)\cap V_{b'}^\delta\big(2\rho^{\frac12}\big)\cap D_4$ is a subset of

$$\big[-c_{14}\rho^{\frac12},c_{14}\rho^{\frac12}\big]\times[-c_{14}\rho^{a+\alpha_d},c_{14}\rho^{a+\alpha_d}]\times([-c_{14}\rho,c_{14}\rho])^{d-3}$$

which has the measure $O\big(\rho^{d-3+\frac12+a+\alpha_d}\big)$. This with $|\Gamma_\delta(\rho^{\alpha_d})|=O(\rho^{(d-1)\alpha_d})$ gives

$$\mu\big(\big(V_b^\delta(3\rho^a)\cap D_1\big(2\rho^{\frac12}\big)\cap D_4\big)=O\big(\rho^{d-3+\frac12+a+d\alpha_d}\big)=o(\rho^{d-2+a}),\tag{115}$$

since $d\alpha_d<\frac12$ (see the definition of α_d in (11)). In the same way, we get

$$\mu\big(V_b^\delta(3\rho^a)\cap D_2(2\rho^{a+2\alpha})\cap D_4\big)=O(\rho^{d-3+2a+(d+4)\alpha})=o(\rho^{d-2+a}),\tag{116}$$

since $a+(d+4)\alpha<1$ (see (11) and (13)). To estimate $\mu(D_{b',j}(\rho,v,6))$, we turn the coordinate axes so that the direction of $(1,0,0,\ldots,0)$ coincides with the direction of b'. Then, the condition $x\in D_{b',j}(\rho,v,6)\cap D_4$ implies that

$$2x_1|b'|+|b'|^2+|(j+v)\delta|^2|=O(\rho^{\alpha_d}),\qquad x_1^2+x_2^2+\cdots+x_{d-1}^2=O(\rho^2).$$

These equalities show that x_1 belongs to the interval of length $O(\rho^{\alpha_d})$ and $\mu(D_{b',j}(\rho,v,6)\cap D_4)=O(\rho^{d-2+\alpha_d})$. Now using (109) and taking into account that $n_1=O(\rho^{\frac12(1+\alpha_d)})$, $|\Gamma_\delta(\rho^{\alpha_d})|=O(\rho^{(d-1)\alpha_d})$, we obtain

$$\mu(D(\rho,v,4)\cap D_4=O(\rho^{d-2+\frac12+(d+\frac12)\alpha_d})=o(\rho^{d-2+a}),$$

since $a>\frac12+(d+\frac12)\alpha_d$ (see (13) and (11)). This estimation with (115), (116) and (113) implies that $\mu(D'(\rho))>c_{15}\rho^{d-2+a}$. Thus, the second equality of (111) follows from (112) .

Now take any element β from $S_2'(\rho,b,v)\cap\Gamma_\delta$. It follows from the definitions of the sets $S_2'(\rho,b,v)$, $D_{b',j}(\rho,v,4)$, $A(\beta,\rho)$ (see (110) and (27)) that $v\notin A(\beta,\rho)$ and (29) hold.

Let us prove the inequalities in (58). By the definition of $S_2'(\rho,b,v)$, we have $\beta\in V_b^\delta(4\rho^a)\big\backslash V_b^\delta(\rho^a)$. This means that $\rho^a\leqslant|2\langle\beta,b\rangle+|b|^2|<4\rho^a$. This with the obvious relations $|b|=O(1)$, $|\tau|=O(1)$ implies (58).

Now we prove (59). If $\gamma\in S(\delta,b)\backslash\delta\mathbb{R}$, then

$$\gamma=nb+a\delta,\qquad n\neq0,\qquad n\in\mathbb{Z},\qquad a\in\mathbb{R},\qquad |\langle\gamma,b\rangle|=|n||b|^2\geqslant|b|^2,\tag{117}$$

since each $\gamma\in\Gamma$ has decomposition $\gamma=b'+a\delta$, where $b'\in\Gamma_\delta$, and b is a visible element of Γ_δ (see (3.2) of [10] and the definition of $S(\delta,b)$ in (5)). This with the relation $\langle\beta+\tau,\delta\rangle=0$ gives $\langle\beta+\tau,\gamma\rangle=n\langle\beta+\tau,b\rangle$. Therefore, the first inequality of (58) implies (59).

Let us prove (60). If $\gamma\notin S(\delta,b)$, $|\gamma|<|\rho|^\alpha$, then $\gamma=b'+a\delta$, where $a\in\mathbb{R}$, $b'\in\Gamma_\delta(\rho^\alpha)\backslash b\mathbb{R}$ and $\langle\beta+\tau,\gamma\rangle=\langle\beta+\tau,b'\rangle$. Therefore, using $|b'|=O(\rho^\alpha)$, $|\tau|=O(1)$ and arguing as in the proof of (58), we see that the relation $\beta\notin V_{b'}^\delta(\rho^{a+2\alpha})$ (see definition of

$S_2'(\rho, b, v))$ implies (60). The inequality (61) follows from the definition of $f_{\delta,\beta+\tau}(x)$, (59), (60) and from the obvious relation

$$\sum_{\gamma\in\Gamma} |\gamma||q_\gamma| < c_{16},$$

(see (1)). The last inequality with (117) implies the convergence of the series (5). □

Appendix A. The proof of (81)

Here, we estimate the conjugate $\overline{C_1(j', \lambda_{j,\beta})}$ of $C_1(j', \lambda_{j,\beta})$, namely, we prove that

$$\sum_{(j_1,\beta_1)\in Q(\rho^\alpha,9r)} \frac{\overline{A(j', \beta, j'+j_1, \beta+\beta_1)}\,\overline{A(j'+j_1, \beta+\beta_1, j, \beta)}}{\lambda_{j,\beta} - \lambda_{j'+j_1,\beta+\beta_1}} = O(\rho^{-2a} r^2), \tag{A.1}$$

(see (79)), where $Q(\rho^\alpha, 9r) = \{(j_1, \beta_1) : |j_1\delta| < 9r, 0 < |\beta_1| < \rho^\alpha\}, j \in S_1(\rho), |j'\delta| < r, r = O\big(\rho^{\frac{1}{2}\alpha_2}\big)$. The conditions on indices j', j_1, j and (39) imply that $\mu_{j'+j_1} = O(r^2), \mu_j = O(r^2)$. These with $\beta \notin V^\delta_{\beta_1}(\rho^a)))$, where $\beta_1 \in \Gamma_\delta(p\rho^\alpha)$, (see (29)) give

$$\lambda_{j,\beta} - \lambda_{j'+j_1,\beta+\beta_1} = -2\langle\beta, \beta_1\rangle + O(r^2), \qquad |\langle\beta, \beta_1\rangle| > \tfrac{1}{3}\rho^a. \tag{A.2}$$

Using this, (34) and (A.1), we get

$$\overline{C_1(j', \lambda_{j,\beta})} = \sum_{\beta_1} \frac{C'}{-2\langle\beta, \beta_1\rangle} + O(\rho^{-2a} r^2), \tag{A.3}$$

where $C' = \sum_{j_1} \overline{A(j', \beta, j'+j_1, \beta+\beta_1)}\,\overline{A(j'+j_1, \beta+\beta_1, j, \beta)}$. In [10], we proved that (see (3.21), (3.7), lemma 3.3 of [10])

$$\begin{aligned}
\overline{A(j', \beta, j'+j_1, \beta+\beta_1)} &= \sum_{n_1:(n_1,\beta_1)\in\Gamma'(\rho^\alpha)} c(n_1, \beta_1) a(n_1, \beta_1, j', \beta, j'+j_1, \beta+\beta_1),\\
\overline{A(j'+j_1, \beta+\beta_1, j, \beta)} &= \sum_{n_2:(n_2,-\beta_1)\in\Gamma'(\rho^\alpha)} c(n_2, -\beta_1) a(n_2, -\beta_1, j'+j_1, \beta+\beta_1, j, \beta),
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
&\Gamma'(\rho^\alpha) = \{(n_1, \beta_1) : \beta_1 \in \Gamma_\delta\backslash 0, n_1 \in \mathbb{Z}, \beta_1 + (n_1 - (2\pi)^{-1}\langle\beta_1, \delta^*\rangle)\delta \in \Gamma(\rho^\alpha)\},\\
&c(n_1, \beta_1) = q_{\gamma_1}, \gamma_1 = \beta_1 + (n_1 - (2\pi)^{-1}\langle\beta_1, \delta^*\rangle)\delta \in \Gamma(\rho^\alpha),
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
a(n_1, \beta_1, j', \beta, j'+j_1, \beta+\beta_1) &= (\mathrm{e}^{\mathrm{i}(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)\zeta}\varphi_{j',v(\beta)}(\zeta), \varphi_{j'+j_1,v(\beta+\beta_1)}(\zeta)),\\
a(n_2, -\beta_1, j'+j_1, \beta+\beta_1, j, \beta) &= (\mathrm{e}^{\mathrm{i}(n_2-(2\pi)^{-1}\langle-\beta_1,\delta^*\rangle)\zeta}\varphi_{j'+j_1,v(\beta+\beta_1)}(\zeta), \varphi_{j,v(\beta)}(\zeta))\\
&= (\varphi_{j'+j_1,v(\beta+\beta_1)}(\zeta), \mathrm{e}^{-\mathrm{i}(n_2-(2\pi)^{-1}\langle-\beta_1,\delta^*\rangle)\zeta}\varphi_{j,v(\beta)}(\zeta))\\
&= \overline{(\mathrm{e}^{-\mathrm{i}(n_2-(2\pi)^{-1}\langle-\beta_1,\delta^*\rangle)\zeta}\varphi_{j,v(\beta)}(\zeta), \varphi_{j'+j_1,v(\beta+\beta_1)}(\zeta))},
\end{aligned} \tag{A.6}$$

where δ^* is the element of Ω satisfying $\langle\delta^*, \delta\rangle = 2\pi$.

Now, to estimate the right-hand side of (A.3) we prove that

$$\begin{aligned}
&\sum_{j_1} a(n_1, \beta_1, j', \beta, j'+j_1, \beta+\beta_1) a(n_2, -\beta_1, j'+j_1, \beta+\beta_1, j, \beta)\\
&\qquad = a(n_1+n_2, 0, j', \beta, j, \beta) + O(\rho^{-p\alpha}).
\end{aligned} \tag{A.7}$$

By definition, we have

$$\begin{aligned}
a(n_1+n_2, 0, j', \beta, j, \beta) &= (\mathrm{e}^{\mathrm{i}(n_1+n_2)\zeta}\varphi_{j',v(\beta)}(\zeta), \varphi_{j,v(\beta)}(\zeta))\\
&= (\mathrm{e}^{\mathrm{i}(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)\zeta}\varphi_{j',v(\beta)}(\zeta), \mathrm{e}^{-\mathrm{i}(n_2-(2\pi)^{-1}\langle-\beta_1,\delta^*\rangle)\zeta}\varphi_{j,v(\beta)}(\zeta)).
\end{aligned}$$

This, (A.6) and the following formulae

$$\mathrm{e}^{\mathrm{i}(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)\zeta}\varphi_{j',v(\beta)}(\zeta) = \sum_{|j_1\delta|<9r} a(n_1,\beta_1,j',\beta,j'+j_{1,}\beta+\beta_1)\varphi_{j'+j_1,v(\beta+\beta_1)}(\zeta)+O(\rho^{-p\alpha}),$$

$$\mathrm{e}^{-\mathrm{i}(n_2-(2\pi)^{-1}\langle-\beta_1,\delta^*\rangle)\zeta}\varphi_{j,v(\beta)}(\zeta) = \sum_{|j_1\delta|<9r} \overline{a(n_2,-\beta_1,j',\beta,j'+j_{1,}\beta+\beta_1)}\varphi_{j'+j_1,v(\beta+\beta_1)}+O(\rho^{-p\alpha}),$$

$$\times\sum_{j_1}|a(n_1,\beta_1,j',\beta,j'+j_{1,}\beta+\beta_1)| = O(1) \tag{A.8}$$

(see (3.16), (3.17) of [10]) give the proof of (A.7). Now from (A.7), (A.4), (A.3) we obtain

$$C' = \sum_{n_1}\sum_{n_2}(c(n_1,\beta_1)c(n_2,-\beta_1)a(n_1+n_2,0,j',\beta,j,\beta)+O(\rho^{-p\alpha})),$$

$$\overline{C_1(j',\lambda_{j,\beta})} = \sum_{\beta_1}\sum_{n_1}\sum_{n_2}C_1'(\beta_1,n_1,n_2)+O(\rho^{-2a}r^2),$$

where $C_1'(\beta_1,n_1,n_2) = \frac{c(n_1,\beta_1)c(n_2,-\beta_1)a(n_1+n_2,0,j',\beta,j,\beta)}{-2\langle\beta,\beta_1\rangle}$. It is clear that

$$C_1'(\beta_1,n_1,n_2)+C_1'(-\beta_1,n_2,n_1) = 0. \tag{A.9}$$

Therefore $\overline{C_1(j',\lambda_{j,\beta})} = O(\rho^{-2a}r^2)$.

Appendix B. The proof of (47)

Arguing as in the proof of (80), we see that $C_2(\Lambda_{j,\beta}) = C_2(\lambda_{j,\beta})+O(\rho^{-3a})$ and by (A.4)

$$\overline{C_2(\lambda_{j,\beta})} = \sum_{\beta_1,\beta_2}\left(\sum_{n_1,n_2,n_3}\left(\sum_{j_1,j_2}\frac{c(n_1,\beta_1)c(n_2,\beta_2)c(n_3,-\beta_1-\beta_2)}{(\lambda_{j,\beta}-\lambda_{j(1),\beta(1)})(\lambda_{j,\beta}-\lambda_{j(2),\beta(2)})}a(n_1,\beta_1,j,\beta,j(1)_{,}\beta(1))\right.\right.$$
$$\times\, a(n_2,\beta_2,j(1)_{,}\beta(1),j(2)_{,}\beta(2))a(n_3,-\beta_1-\beta_2,j(2),\beta(2),j,\beta),$$

where $(j_1,\beta_1)\in Q(\rho^\alpha,9r_1)$, $(j_2,\beta_2)\in Q(\rho^\alpha,90r_1)$, $j\in S_1$, $\beta_1+\beta_2\neq 0$. Applying (A.7) two times and using (A.8), we get

$$\sum_{j_1}a(n_1,\beta_1,j,\beta,j(1)_{,}\beta(1))$$
$$\times\left(\sum_{j_2}a(n_2,\beta_2,j(1)_{,}\beta(1),j(2)_{,}\beta(2))a(n_3,-\beta_1-\beta_2,j(2)_{,}\beta(2),j,\beta)\right)$$
$$= \sum_{j_1}a(n_1,\beta_1,j,\beta,j(1)_{,}\beta(1))(a(n_2+n_3,-\beta_1,j(1)_{,}\beta(1),j,\beta)+O(\rho^{-p\alpha}))$$
$$= a(n_1+n_2+n_3,0,j,\beta,j,\beta)+O(\rho^{-p\alpha}).$$

Using this in the above expression for $C_2(\lambda_{j,\beta})$ and taking into account that

$$\lambda_{j,\beta}-\lambda_{j(1)_,\beta(1)} = -2\langle\beta,\beta_1\rangle+O(\rho^{2\alpha_1}), \qquad |\langle\beta,\beta_1\rangle| > \tfrac{1}{3}\rho^a,$$

$$\lambda_{j,\beta}-\lambda_{j(2)_,\beta(2)} = -2\langle\beta,\beta_1+\beta_2\rangle+O(\rho^{2\alpha_1}), \qquad |\langle\beta,\beta_1+\beta_2\rangle| > \tfrac{1}{3}\rho^a,$$

which can be proved as (A.2), we have $C_2(\lambda_{j,\beta}) = O(\rho^{-3a+2\alpha_1})+$

$$\sum_{\beta_1,\beta_2}\sum_{n_1,n_2,n_3}\frac{c(n_1,\beta_1)c(n_2,\beta_2)c(n_3,-\beta_1-\beta_2)a(n_1+n_2+n_3,0,j,\beta,j,\beta)}{4\langle\beta,\beta_1\rangle\langle\beta,\beta_1+\beta_2\rangle}.$$

Grouping the terms with the equal multiplicands

$c(n_1,\beta_1)c(n_2,\beta_2)c(n_3,-\beta_1-\beta_2)$, $c(n_2,\beta_2)c(n_1,\beta_1)c(n_3,-\beta_1-\beta_2)$,
$c(n_1,\beta_1)c(n_3,-\beta_1-\beta_2)c(n_2,\beta_2)$, $c(n_2,\beta_2)c(n_3,-\beta_1-\beta_2)c(n_1,\beta_1)$,
$c(n_3,-\beta_1-\beta_2)c(n_1,\beta_1)c(n_2,\beta_2)$, $c(n_3,-\beta_1-\beta_2)c(n_2,\beta_2)c(n_1,\beta_1)$

and using the obvious equality

$$\frac{1}{\langle\beta,\beta_1\rangle\langle\beta,\beta_1+\beta_2\rangle}+\frac{1}{\langle\beta,\beta_2\rangle\langle\beta,\beta_2+\beta_1\rangle}+\frac{1}{\langle\beta,\beta_1\rangle\langle\beta,-\beta_2\rangle}$$
$$+\frac{1}{\langle\beta,\beta_2\rangle\langle\beta-,\beta_1\rangle}+\frac{1}{\langle\beta,-\beta_1-\beta_2\rangle\langle\beta,-\beta_2\rangle}+\frac{1}{\langle\beta,-\beta_1-\beta_2\rangle\langle\beta,-\beta_1\rangle}=0,$$

we see that $C_2(\lambda_{j,\beta})=O(\rho^{-3a+2\alpha_1})$.

Appendix C. The proof of (46)

By (80), we have $C_1(\Lambda_{j,\beta})=C_1(\lambda_{j,\beta})+O(\rho^{-3a})$. Therefore, we need to prove that

$$\overline{C_1(\lambda_{j,\beta})}=\frac{1}{4}\int_F|f_{\delta,\beta+\tau}(x)|^2|\varphi_{j,v}^{\delta}(\langle\delta,x\rangle)|^2\,\mathrm{d}x+O(\rho^{-3a+2\alpha_1}),$$

where

$$\overline{C_1(\lambda_{j,\beta})}\equiv\sum_{\beta_1}\sum_{j_1}\frac{\overline{A(j,\beta,j+j_1,\beta+\beta_1)}\,\overline{A(j+j_1,\beta+\beta_1,j,\beta)}}{\lambda_{j,\beta}-\lambda_{j+j_1,\beta+\beta_1}},$$

$(j_1,\beta_1)\in Q(\rho^\alpha,9r_1)$, $j\in S_1$, and by (A.4)

$$\overline{C_1(\lambda_{j,\beta})}=\sum_{\beta_1}\sum_{n_1:(n_1,\beta_1)\in\Gamma'(\rho^\alpha)}\sum_{n_2:(n_2,-\beta_1)\in\Gamma'(\rho^\alpha)}\sum_{j_1}\frac{c(n_1,\beta_1)c(n_2,-\beta_1)}{\lambda_{j,\beta}-\lambda_{j+j_1,\beta+\beta_1}}$$
$$\times a(n_1,\beta_1,j,\beta,j+j_1,\beta+\beta_1)a(n_2,-\beta_1,j+j_1,\beta+\beta_1,j,\beta).$$

Replacing $\lambda_{j,\beta}-\lambda_{j+j_1,\beta+\beta_1}$ by $-(2\langle\beta+\tau,\beta_1\rangle+|\beta_1|^2+\mu_{j+j_1}(v(\beta+\beta_1))-\mu_j(v(\beta)))$ and using (A.7) for $j'=j$, we have

$$\overline{C_1(j,\lambda_{j,\beta})}=\sum_{\beta_1}\sum_{n_1}\sum_{n_2}\frac{c(n_1,\beta_1)c(n_2,-\beta_1)a(n_1+n_2,0,j,\beta,j,\beta)}{-2\langle\beta+\tau,\beta_1\rangle}$$
$$+\sum_{\beta_1}\sum_{n_1}\sum_{n_2}\sum_{j_1}\frac{c(n_1,\beta_1)c(n_2,-\beta_1)a(n_1,\beta_1,j,\beta,j+j_1,\beta+\beta_1)}{2\langle\beta+\tau,\beta_1\rangle(2\langle\beta+\tau,\beta_1\rangle+|\beta_1|^2+\mu_{j+j_1}-\mu_j)}$$
$$\times a(n_2,-\beta_1,j+j_1,\beta+\beta_1,j,\beta)(|\beta_1|^2+\mu_{j+j_1}(v(\beta+\beta_1))-\mu_j(v(\beta))).$$

The formula (A.9) shows that the first summation of the right-hand side of this equality is zero. Thus, we need to estimate the second sum. For this, we use the following relation

$$\mu_{j+j_1}(v(\beta+\beta_1))a(n_1,\beta_1,j,\beta,j+j_1,\beta+\beta_1)=\left(\mathrm{e}^{\mathrm{i}(n_1-\frac{\langle\beta_1,\delta^*\rangle}{2\pi})\zeta}\varphi_{j,v(\beta)}(\zeta),T_v\varphi_{j+j_1,v(\beta+\beta_1)}(\zeta)\right)$$
$$=(T_v(\mathrm{e}^{\mathrm{i}(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)\zeta}\varphi_{j,v(\beta)}(\zeta)),\varphi_{j+j_1,v(\beta+\beta_1)}(\zeta)$$
$$=(|n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle|^2|\delta|^2+\mu_j(v))(\mathrm{e}^{\mathrm{i}(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)\zeta}\varphi_{j,v(\beta)}(\zeta),\varphi_{j+j_1,v(\beta+\beta_1)}(\zeta))$$
$$-2\mathrm{i}(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)|\delta|^2(\mathrm{e}^{\mathrm{i}(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)\zeta}\varphi'_{j,v(\beta)}(\zeta)),\varphi_{j+j_1,v(\beta+\beta_1)}(\zeta)).$$

Using this, (A.7) and the formula

$$\sum_{j_1}(\mathrm{e}^{\mathrm{i}(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)\zeta}\varphi'_{j,v(\beta)}(\zeta)),\varphi_{j+j_1,v(\beta+\beta_1)}(\zeta))a(n_2,-\beta_1,j+j_1,\beta+\beta_1,j,\beta)$$
$$=(\mathrm{e}^{\mathrm{i}(n_1+n_2)\zeta}\varphi'_{j,v(\beta)}(\zeta)),\varphi_{j,v(\beta)}(\zeta))+O(\rho^{-p\alpha})$$

which can be proved as (A.7), we obtain

$$\sum_{j_1}\mu_{j+j_1}(v(\beta+\beta_1))a(n_1,\beta_1,j,\beta,j+j_1,\beta+\beta_1)a(n_2,-\beta_1,j+j_1,\beta+\beta_1,j,\beta)$$
$$=(|n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle|^2)|\delta|^2+\mu_j(v))a(n_1+n_2,0,j,\beta,j,\beta)$$
$$-2\mathrm{i}(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)|\delta|^2(\mathrm{e}^{\mathrm{i}(n_1+n_2)\zeta}\varphi'_{j,v(\beta)}(\zeta)),\varphi_{j,v(\beta)}(\zeta)). \qquad \text{(C.1)}$$

Here, the last multiplicand can be estimated as follows

$$\mu_j(v)(\varphi_{j,v(\beta)}(\zeta),\mathrm{e}^{\mathrm{i}(n_1+n_2)\zeta}\varphi_{j,v(\beta)}(\zeta))=(\varphi_{j,v(\beta)}(\zeta),T_v(\mathrm{e}^{\mathrm{i}(n_1+n_2)\zeta}\varphi_{j,v(\beta)}(\zeta)))$$
$$=(n_1+n_2)^2|\delta|^2(\varphi_{j,v(\beta)}(\zeta),\mathrm{e}^{\mathrm{i}(n_1+n_2)\zeta}\varphi_{j,v(\beta)}(\zeta))$$
$$+2\mathrm{i}(n_1+n_2)|\delta|^2(\varphi_{j,v(\beta)}(\zeta),\mathrm{e}^{\mathrm{i}(n_1+n_2)\zeta}\varphi'_{j,v(\beta)}(\zeta))+\mu_j(v)(\varphi_{j,v(\beta)},\mathrm{e}^{\mathrm{i}(n_1+n_2)\zeta}\varphi_{j,v(\beta)}),$$
$$\times(\mathrm{e}^{\mathrm{i}(n_1+n_2)\zeta}\varphi'_{j,v(\beta)}(\zeta)),\varphi_{j,v(\beta)}(\zeta))=\frac{n_1+n_2}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}(n_1+n_2)\zeta}\varphi_{j,v(\beta)}(\zeta)),\varphi_{j,v(\beta)}(\zeta)).$$

Using this, (C.1), and (A.7), we get

$$\sum_{j_1}(a(n_1,\beta_1,j,\beta,j+j_{1,}\beta+\beta_1)a(n_2,-\beta_1,j+j_1,\beta+\beta_1,j,\beta))$$
$$\times(|\beta_1|^2+\mu_{j+j_1}(v(\beta+\beta_1))-\mu_j(v(\beta)))=a(n_1+n_2,0,j,\beta,j,\beta)$$
$$\times\left(|\beta_1|^2+|n_1-\frac{\langle\beta_1,\delta^*\rangle}{2\pi}|^2|\delta|^2-\left(n_1-\frac{\langle\beta_1,\delta^*\rangle}{2\pi}\right)|\delta|^2(n_1+n_2)\right)$$
$$=\left(|\beta_1|^2+|\delta|^2\left(n_1-\frac{\langle\beta_1,\delta^*\rangle}{2\pi}\right)\left(-n_2-\frac{\langle\beta_1,\delta^*\rangle}{2\pi}\right)\right)a(n_1+n_2,0,j,\beta,j,\beta).$$

Thus, $\overline{C_1(j,\lambda_{j,\beta})}=C+O(\rho^{-3a+2\alpha_1})$, where

$$C=\sum_{\beta_1,n_1,n_2}\frac{c(n_1,\beta_1)c(n_2,-\beta_1)a(n_1+n_2,0,j,\beta,j,\beta)}{4|\langle\beta+\tau,\beta_1\rangle|^2}$$
$$\times\left(|\beta_1|^2+\left(n_1-\frac{\langle\beta_1,\delta^*\rangle}{2\pi}\right)\left(-n_2-\frac{\langle\beta_1,\delta^*\rangle}{2\pi}\right)|\delta|^2\right). \qquad \text{(C.2)}$$

Now we consider $\int_F|f_{\delta,\beta+\tau}(x)|^2|\varphi_{n,v}(\langle\delta,x\rangle)|^2\,\mathrm{d}x$, where $f_{\delta,\beta+\tau}$ is defined in (13). By (A.5),

$$f_{\delta,\beta+\tau}(x)=\sum_{(n_1,\beta_1)\in\Gamma'_\delta(\rho^\alpha)}\frac{\beta_1+\left(n_1-\frac{\langle\beta_1,\delta^*\rangle}{2\pi}\right)\delta}{\langle\beta+\tau,\beta_1\rangle}c(n_1,\beta_1)\,\mathrm{e}^{\mathrm{i}\langle\beta_1+(n_1-\frac{\langle\beta_1,\delta^*\rangle}{2\pi})\delta,x\rangle}.$$

Here $f_{\delta,\beta+\tau}(x)$ is a vector of $\mathbb{R}^d$. Using $\langle\beta,\delta\rangle=0$ for $\beta\in\Gamma_\delta$, we obtain

$$|f_{\delta,\beta+\tau}(x)|^2=\sum_{(n_1,\beta_1),(n_2,\beta_2)\in\Gamma'_\delta(\rho^\alpha)}\frac{\langle\beta_1,\beta_2\rangle+(n_1-\frac{\langle\beta_1,\delta^*\rangle}{2\pi})(n_2-\frac{\langle\beta_1,\delta^*\rangle}{2\pi})|\delta|^2}{\langle\beta+\tau,\beta_1\rangle\langle\beta+\tau,\beta_2\rangle}$$
$$\times c(n_1,\beta_1)c(-n_2,-\beta_2)\,\mathrm{e}^{\mathrm{i}\langle\beta_1-\beta_2+(n_1-n_2-(2\pi)^{-1}\langle\beta_1-\beta_2,\delta^*\rangle)\delta,x\rangle}.$$

Since $\varphi_{j,v}(\langle\delta,x\rangle)$ is a function of $\langle\delta,x\rangle$, we have

$$\int_F\mathrm{e}^{\mathrm{i}\langle\beta_1-\beta_2+(n_1-n_2-(2\pi)^{-1}\langle\beta_1-\beta_2,\delta^*\rangle)\delta,x\rangle}|\varphi_{j,v}(\langle\delta,x\rangle)|^2\,\mathrm{d}x=0$$

for $\beta_1\neq\beta_2$. Therefore,

$$\int_F|f_{\delta,\beta+\tau}(x)|^2|\varphi_{j,v}(\langle\delta,x\rangle)|^2\,\mathrm{d}x=\sum_{\beta_1,n_1,n_2}\frac{c(n_1,\beta_1)c(-n_2,-\beta_1)}{|\langle\beta+\tau,\beta_1\rangle|^2}$$
$$\times\left(|\beta_1|^2+\left(n_1-\frac{\langle\beta_1,\delta^*\rangle}{2\pi}\right)\left(n_2-\frac{\langle\beta_1,\delta^*\rangle}{2\pi}\right)|\delta|^2a(n_1-n_2,0,j,\beta,j,\beta\right\rangle.$$

Replacing n_2 by $-n_2$, we get $\int_F|f_{\delta,\beta+\tau}(x)|^2|\varphi_{n,v}(\langle\delta,x\rangle)|^2\,\mathrm{d}x=4C$ (see (C.2)) and (46).

Appendix D. Asymptotic formulae for $T_v(Q)$

It is well-known that the large eigenvalues of $T_0(Q)$ lie in $O\left(\frac{1}{m^4}\right)$ neighborhood of

$$|m\delta| + \frac{1}{16\pi|m\delta|^3}\int_0^{2\pi}|Q(t)|^2\,\mathrm{d}t$$

for the large values of m (see [1], p 58). This formula yields the invariant (17). Using the asymptotic formulae for solutions of the Sturm–Liouville equation (see [1], p 63), one can easily obtain that

$$\varphi_{n,v}(\zeta) = \mathrm{e}^{\mathrm{i}(n+v)\zeta}\left(\left(1 + \frac{Q_1(\zeta)}{2\mathrm{i}(n+v)|\delta|^2} + \frac{Q(\zeta) - Q(0) - \frac{1}{2}Q_1^2(\zeta)}{4(n+v)^2|\delta|^4}\right) + O\left(\frac{1}{n^3}\right)\right),$$

where $Q_1(\zeta) = \int_0^{\zeta} Q(t)\,\mathrm{d}t$. From this, by direct calculations, we find $A_0(\zeta), A_1(\zeta), A_2(\zeta)$ (see (6)) and then using these in (7), we get the invariant (16).

Now we consider the eigenfunction $\varphi_{n,v}(\zeta)$ of $T_v(p)$ in the case $v \neq 0, \frac{1}{2}$ and $p(\zeta) = p_1\,\mathrm{e}^{\mathrm{i}\zeta} + p_{-1}\,\mathrm{e}^{-\mathrm{i}\zeta}$. The eigenvalues and the eigenfunctions of $T_v(0)$ are $(n+v)^2|\delta|^2$ and $\mathrm{e}^{\mathrm{i}(n+v)\zeta}$, for $n \in \mathbb{Z}$. Since the eigenvalues of $T_v(p)$ are simple for $v \neq 0, \frac{1}{2}$, by the well-known perturbation formula

$$(\varphi_{n,v}(\zeta), \mathrm{e}^{\mathrm{i}(n+v)\zeta})\varphi_{n,v}(\zeta) = \mathrm{e}^{\mathrm{i}(n+v)\zeta} + \sum_{k=1,2,\ldots}\frac{(-1)^{k+1}}{2\mathrm{i}\pi}\int_C (T_v(0)-\lambda)^{-1}p(x))^k(T_v(0)-\lambda)^{-1}\,\mathrm{e}^{\mathrm{i}(n+v)\zeta}\,\mathrm{d}\lambda, \qquad \text{(D.1)}$$

where C is a contour containing only the eigenvalue $(n+t)^2|\delta|^2$. Using

$$(T_v(0)-\lambda)^{-1}\,\mathrm{e}^{\mathrm{i}(n+v)\zeta} = \frac{\mathrm{e}^{\mathrm{i}(n+v)\zeta}}{(n+v)^2|\delta|^2 - \lambda},$$

we see that the kth ($k = 1, 2, 3, 4$) term F_k of the series (D.1) has the form

$$F_1 = \frac{1}{2\mathrm{i}\pi}\int_C \sum_{m=1,-1}\frac{p_m\,\mathrm{e}^{\mathrm{i}(n+m+v)\zeta}}{((n+v)^2|\delta|^2-\lambda)((n+m+v)^2|\delta|^2-\lambda)}\,\mathrm{d}\lambda,$$

$$F_2 = \frac{-1}{2\mathrm{i}\pi}\int_C \sum_{m,l=1,-1}\frac{p_m p_l\,\mathrm{e}^{\mathrm{i}(n+m+l+v)\zeta}}{((n+v)^2|\delta|^2-\lambda)} \times \frac{1}{((n+m+v)^2|\delta|^2-\lambda)((n+m+l+v)^2|\delta|^2-\lambda)}\,\mathrm{d}\lambda,$$

$$F_3 = \frac{1}{2\mathrm{i}\pi}\int_C \sum_{m,l,k=1,-1}\frac{p_m p_l p_k\,\mathrm{e}^{\mathrm{i}(n+m+l+k+v)\zeta}}{((n+v)^2|\delta|^2-\lambda)((n+m+v)^2|\delta|^2-\lambda)} \times \frac{1}{((n+m+l+v)^2|\delta|^2-\lambda)((n+m+l+k+v)^2|\delta|^2-\lambda)}\,\mathrm{d}\lambda,$$

$$F_4 = \frac{-1}{2\mathrm{i}\pi}\int_C \sum_{m,l,k,r=1,-1}\frac{p_m p_l p_k p_r\,\mathrm{e}^{\mathrm{i}(n+m+l+k+r+v)\zeta}}{((n+m+l+k+r+v)^2|\delta|^2-\lambda)} \times \frac{1}{((n+m+v)^2|\delta|^2-\lambda)((n+m+l+v)^2|\delta|^2-\lambda)} \times \frac{1}{((n+m+l+k+v)^2|\delta|^2-\lambda)((n+v)^2|\delta|^2-\lambda)}\,\mathrm{d}\lambda.$$

Since the distance between $(n+v)^2|\delta|^2$ and $(n'+v)^2|\delta|^2$ for $n' \neq n$ is greater than $c_{17}n$, we can choose the contour C such that

$$\frac{1}{|(n'+v)^2|\delta|^2 - \lambda|} < \frac{c_{18}}{n}, \qquad \forall \lambda \in C, \qquad \forall n' \neq n$$

and the length of C is less than c_{19}. Therefore $F_5 + F_6 + \cdots = O(n^{-5})$. Now, we calculate the integrals in F_1, F_2, F_3, F_4 by the Cauchy integral formula and then decompose the obtained expression in powers of $\frac{1}{n}$. Then,

$$F_1 = \mathrm{e}^{\mathrm{i}(n+v)\zeta}\left((p_1\,\mathrm{e}^{\mathrm{i}\zeta} - p_{-1}\,\mathrm{e}^{-\mathrm{i}\zeta})\frac{1}{|\delta|^2}\left(\frac{-1}{2n} + \frac{v}{2n^2} - \frac{4v^2+1}{8n^3} + O\left(\frac{1}{n^4}\right)\right)\right.$$
$$\left. + (p_1\,\mathrm{e}^{\mathrm{i}\zeta} + p_{-1}\,\mathrm{e}^{-\mathrm{i}\zeta})\frac{1}{|\delta|^2}\left(\frac{v}{4n^2} - \frac{v}{2n^3} + \frac{12v^2+1}{16n^4} + O\left(\frac{1}{n^5}\right)\right)\right).$$

Let $F_{2,1}$ and $F_{2,2}$ be the sum of the terms in F_2 for which $m+l=\pm 2$ and $m+l=0$ respectively, i.e., $F_2 = F_{2,1} + F_{2,2}$, where

$$F_{2,1} = \mathrm{e}^{\mathrm{i}(n+v)\zeta}\left(((p_1)^2\,\mathrm{e}^{2\mathrm{i}\zeta} + (p_{-1})^2\,\mathrm{e}^{-2\mathrm{i}\zeta})\frac{1}{|\delta|^4}\left(\frac{-1}{8n^2} + \frac{-v}{4n^3} - \frac{12v^2+7}{32n^4} + O\left(\frac{1}{n^5}\right)\right)\right.$$
$$\left. + ((p_1)^2\,\mathrm{e}^{2\mathrm{i}\zeta} - (p_{-1})^2\,\mathrm{e}^{-2\mathrm{i}\zeta})\frac{1}{|\delta|^4}\left(\frac{-3}{16n^3} + O(\frac{1}{n^4})\right)\right),$$
$$F_{2,2} = \mathrm{e}^{\mathrm{i}(n+v)\zeta}|p_1|^2\left(\frac{c_{20}}{n^2} + +\frac{c_{21}}{n^3} + \frac{c_{22}}{n^4} + O\left(\frac{1}{n^5}\right)\right)$$

and c_{20}, c_{21}, c_{22} are the known constants. Similarly, $F_3 = F_{3,1} + F_{3,2}$, where $F_{3,1}$ and $F_{3,2}$ are the sum of the terms in F_3 for which $m+l+k=\pm 3$ and $m+l+k=\pm 1$, respectively. Hence

$$F_{3,1} = \mathrm{e}^{\mathrm{i}(n+v)\zeta}\left(\left(p_1^3\,\mathrm{e}^{3i\zeta} - p_{-1}^3\,\mathrm{e}^{-\mathrm{i}\zeta}\right)\frac{1}{|\delta|^6}\left(\frac{-1}{48n^3} + O\left(\frac{1}{n^4}\right)\right)\right.$$
$$\left. + \left(p_1^3\,\mathrm{e}^{3i\zeta} + p_{-1}^3\,\mathrm{e}^{-3i\zeta}\right)\frac{1}{|\delta|^6}\left(\frac{1}{16n^4} + O\left(\frac{1}{n^5}\right)\right)\right),$$
$$F_{3,2} = \mathrm{e}^{\mathrm{i}(n+v)\zeta}\left((p_1\,\mathrm{e}^{\mathrm{i}\zeta} - p_{-1}\,\mathrm{e}^{-\mathrm{i}\zeta})|p_1|^2\left(\frac{c_{23}}{n^3} + \frac{c_{24}}{n^4} + O\left(\frac{1}{n^5}\right)\right)\right.$$
$$\left. + (p_1\,\mathrm{e}^{\mathrm{i}\zeta} + p_{-1}\,\mathrm{e}^{-\mathrm{i}\zeta})|p_1|^2\left(\frac{c_{25}}{n^4} + O\left(\frac{1}{n^5}\right)\right)\right).$$

Similarly $F_4 = F_{4,1} + F_{4,2} + F_{4,3}$, where $F_{4,1}, F_{4,2}, F_{4,3}$ are the sum of the terms in F_4 for which $m+l+k+r=\pm 4$, $m+l+k+r=\pm 2$, $m+l+k+r=0$, respectively. Thus

$$F_{4,1} = \mathrm{e}^{\mathrm{i}(n+v)\zeta}\left(p_1^4\,\mathrm{e}^{4\mathrm{i}\zeta} + p_{-1}^4\,\mathrm{e}^{-4\mathrm{i}\zeta}\right)\frac{1}{|\delta|^8}\left(\frac{1}{384n^4} + O\left(\frac{1}{n^5}\right)\right),$$
$$F_{4,2} = \mathrm{e}^{\mathrm{i}(n+v)\zeta}\left(p_1^2\,\mathrm{e}^{2\mathrm{i}\zeta} + p_{-1}^2\,\mathrm{e}^{-2\mathrm{i}\zeta}\right)|p_1|^2\left(\frac{c_{26}}{n^4} + O\left(\frac{1}{n^5}\right)\right)),$$
$$F_{4,3} = \mathrm{e}^{\mathrm{i}(n+v)\zeta}|p_1|^4\left(\frac{c_{27}}{n^4} + O\left(\frac{1}{n^5}\right)\right)).$$

Since $p_{-1}^k\,\mathrm{e}^{-\mathrm{i}k\zeta}$ is the conjugate of $p_1^k\,\mathrm{e}^{\mathrm{i}k\zeta}$, the real and imaginary parts of $F_k\,\mathrm{e}^{-\mathrm{i}(n+v)\zeta}$ consist of terms with multiplicands $p_1^k\,\mathrm{e}^{ik\zeta} + p_{-1}^k\,\mathrm{e}^{-ki\zeta}$ and $p_1^k\,\mathrm{e}^{ik\zeta} - p_{-1}^k\,\mathrm{e}^{-ik\zeta}$, respectively. Taking into account this and using the above estimations, we get

$$|(\varphi_{n,v},\mathrm{e}^{\mathrm{i}(n+v)\zeta})\varphi_{n,v}|^2$$

$$= 2\left(\sum_{k=1,2,3,4}\mathrm{Re}(F_k)+\mathrm{Re}(F_1F_2)+\mathrm{Re}(F_1F_3)\right)+|F_1|^2+|F_2|^2+O(n^{-5})$$

$$= 1+\frac{1}{2n^2}\frac{1}{|\delta|^2}(p_1\,\mathrm{e}^{\mathrm{i}\zeta}+p_{-1}\,\mathrm{e}^{-\mathrm{i}\zeta}+c_{28}|p_1|^2)+\frac{1}{n^3}((p_1\,\mathrm{e}^{\mathrm{i}\zeta}+p_{-1}\,\mathrm{e}^{-\mathrm{i}\zeta})c_{29}$$

$$+c_{30}|p_1|^2)+\frac{1}{n^4}\big((p_1\,\mathrm{e}^{\mathrm{i}\zeta}+p_{-1}\,\mathrm{e}^{-\mathrm{i}\zeta})c_{31}+c_{32}|p_1|^2+c_{33}|p_1|^4$$

$$+c_{34}|p_1|^2(p_1\,\mathrm{e}^{\mathrm{i}\zeta}+p_{-1}\,\mathrm{e}^{-\mathrm{i}\zeta})+(c_{35}+c_{36}|p_1|^2)(p_1^2\,\mathrm{e}^{2\mathrm{i}\zeta}+p_{-1}^2\,\mathrm{e}^{-2\mathrm{i}\zeta})\big)+O\left(\frac{1}{n^5}\right),$$

where $\mathrm{Re}(F)$ denotes the real part of F. On the other hand

$$|(\varphi_{n,v}(\zeta),\mathrm{e}^{\mathrm{i}(n+v)\zeta})|^2=\left(c_{37}\frac{1}{n^2}+c_{38}\frac{1}{n^3}+c_{39}\frac{1}{n^4}\right)|p_1|^2+c_{40}\frac{1}{n^4}|p_1|^4+O\left(\frac{1}{n^5}\right).$$

These equalities imply (19). The equality (20) is a consequence of (19), (17) and (7) for $k=2,4$.

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